Lesson 7: Inscribed angles

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Theorem 1.
Opposite angles of cyclic quadrilaterals sum to 180°.

Theorem 2.
Conversely, any quadrilateral for which this is true can be inscribed in a circle.

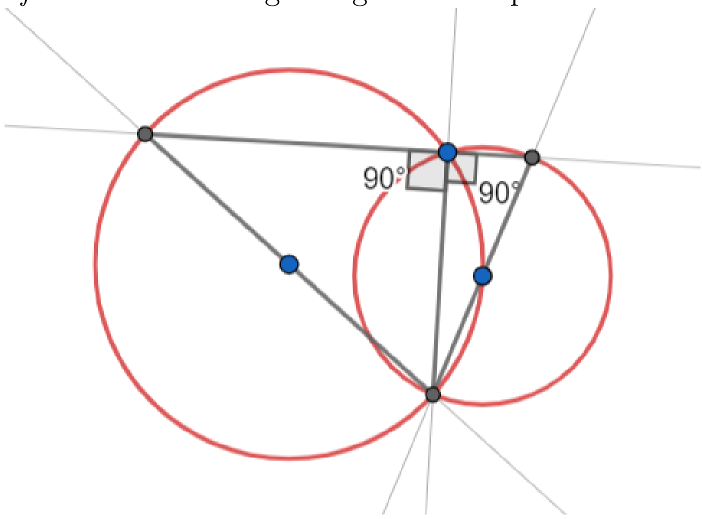
Theorem 3.
If $ABCD$ is a convex quadrilateral with $\angle ACB = \angle ADB$, then it is cyclic.

Problem 0.
Let D be a point on a circumcircle of the equilateral triangle ABC. Prove that 2 of the three angles ADB, BDC and CDA are equal.

Problem 1.
The chords AB and CD intersect at a point M lying inside the circle. Prove that the triangles AMD and CMB are similar.

Problem 2.
Through one of the two intersection points of two circles, a diameter in each of the circles is drawn. Prove that the line connecting the endpoints of these diameters passes through the other intersection point.

Proof. Note the two right angles on the picture:



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Problem 3.
Triangle ABC is acute. Prove that its circumscribed circle, bisector of angle A and the perpendicular bisector of BC all intersect at one point.

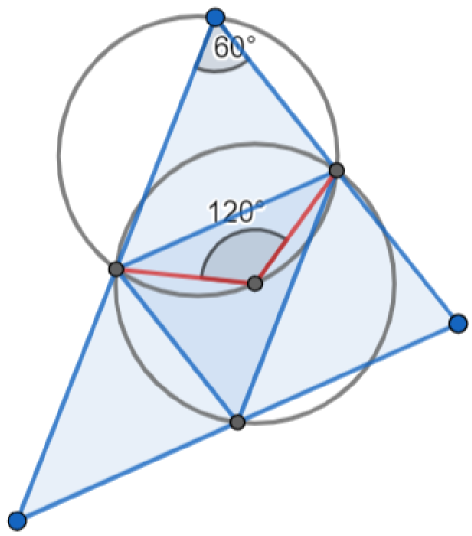
Proof. Take the middle point M of the shorter arc BC. Then both angle bisector and perpendicular bisector pass through it by "equal angles stand on equal arcs or chords" theorem. □

Problem 4.
The angle at the vertex A of a triangle ABC is $60°$. Prove that the center of the circle passing through the midpoints of the sides of triangle ABC, lies on the bisector of angle A.

Proof. Denote the midpoint of AB by C', the midpoint of BC by A' and the midpoint of AC by B'. Then by the middle line theorem and by SSS we have all the triangles $A'B'C'$, $A'BC$, $AB'C$ and ABC' equal to each other.

Let O be the center of the circle passing through A', B' and C'. By inscribed angle theorem we have $\angle B'OC' = 2\angle B'A'C' = 2\angle B'AC' = 120°$. So $B'AC'O$ is a cyclic quadrilateral.

$OB' = OC'$, as they are radii of one circle. So they subtend equal arcs in another circle and $\angle B'AO = C'AO$. So AO is a bisector of A. (Or, alternatively, you can use the previous problem)



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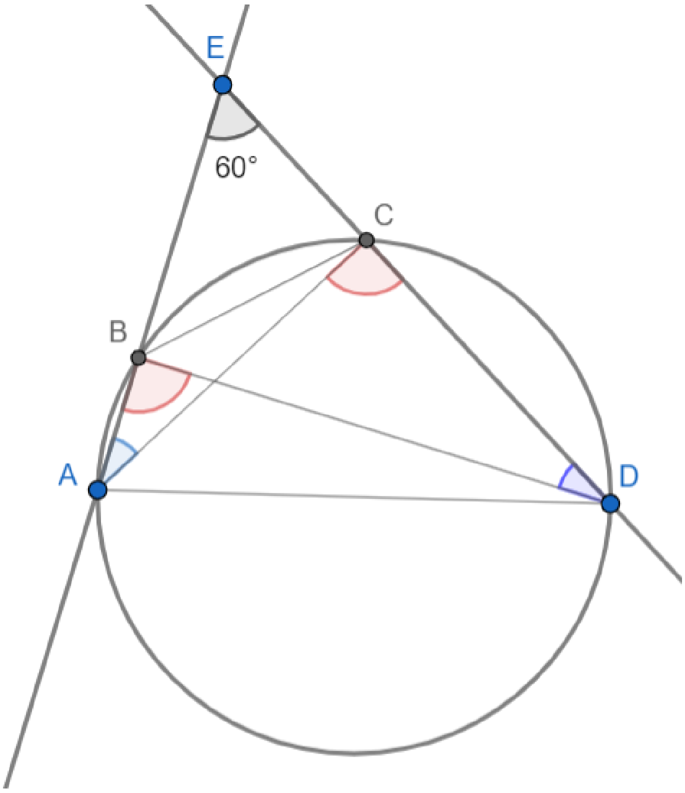
Problem 5.
Four points of the circle follow in order: A, B, C, D. The extensions of the AB chord beyond the point B and the CD chord beyond the point C intersect at the point E. The angle AED is $60°$. The angle ABD is three times greater than the angle BAC. Prove that AD is a diameter of the circle.

Proof. $ABCD$ is cyclic, so we have the bunch of angle equalities including two with the angles mentioned in the statement, namely $\angle ABD = \angle ACD$ and $\angle BDC = \angle BAC = \frac{\angle ABD}{3}$. The latter we will use in the sequence of equalities.

$\angle ABD$ is the outer angle of the triangle DBE. So $\angle ABD = \angle BDE + \angle DEB = \angle BDC + 60° = \frac{\angle ABD}{3} + 60°$.

So $\frac{2}{3}\angle ABD = 60°$, which gives us $\angle ABD = 90°$. Then AD is a diameter.



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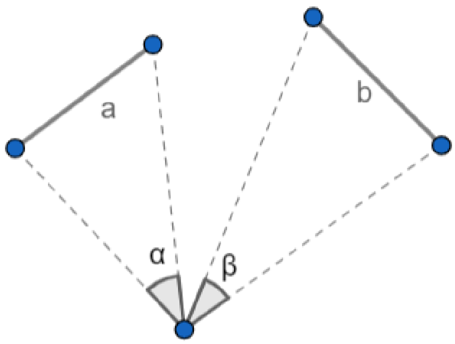
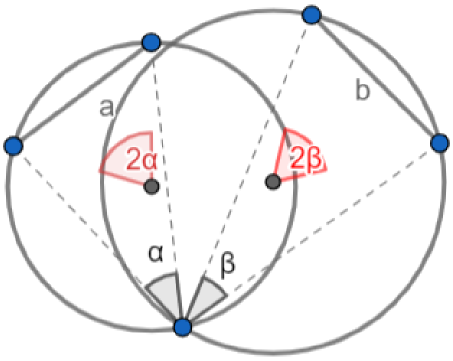
Problem 6.
Two segments a and b are given in the plane. Using a compass and a straightedge, draw a point from which one sees segment a under a given angle α, and segment b at a given angle β.



Proof. It is an intersection of two circles!



Note that here we have 2 solutions, but we can also have 1 or 0 solutions if the circles touch or never intersect. (Also here we don't care about angle orientation which makes this problem even worse)

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Problem 7.
Prove that given a triangle ABC and a point P on its circumcircle, the three closest points to P on lines AB, AC, and BC are collinear.

Proof. This line is called Simson line (it is a link to Wikipedia). □