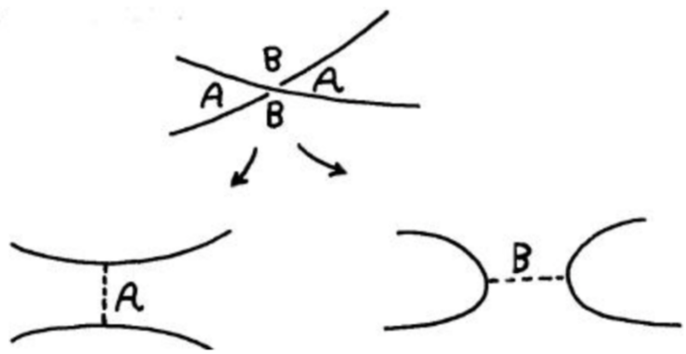
QUANDLES AND KNOT INVARIANTS II

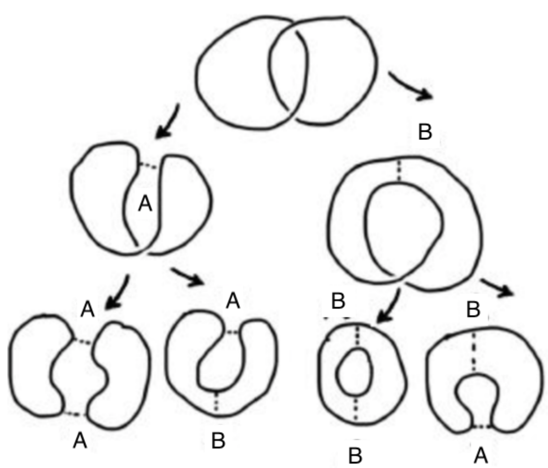
LOS ANGELES MATH CIRCLE
ADVANCED 2
MAY 17, 2020

1. Kauffman Bracket

In general, it is difficult to figure out when two knots are not equivalent. The earliest examples of knot invariants, like tricolorability, are not complex enough for many examples. Our goal in this section is to construct a polynomial for each knot diagram that is invariant under Reidemeister moves. We start by cutting our knot (or link) K into simple pieces. At each crossing in our knot diagram, we make two new descendant links by performing a type-A or type-B splicing:



We keep track of each splicing and its type by labeling it A or B. Continue this way until we reach the **primitive descendants**, the descendants with no crossings. For example, we calculate the four primitive descendants of the "Hopf link":



In the following, we will consider A and B as commuting variables. Define the **Kauffman bracket** $\langle K \rangle$ as follows:

1. If P is a primitive descendant of K, define the bracket $\langle K|P \rangle$ as the product of all labels of P. For example, the leftmost primitive descendant of the Hopf link has bracket $\langle K|P \rangle = B^2$.
2. Let $||P||$ be the number of components (disjoint knots) in P, minus 1.

(3) Define

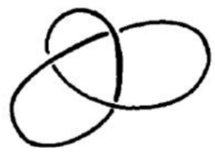
$$\langle K\rangle = \sum_P \langle K|P\rangle d^{||P||}$$

where the sum is over all primitive descendants P and d is another variable.

Problem 1. Find the Kauffman bracket of the Hopf link given the decomposition above.

Problem 2. Let K be a knot. What is $\langle \circ K\rangle$, where $\circ K$ is the knot formed by adding a disjoint unknot to K?

Problem 3. Let T be the trefoil knot shown below. Compute $\langle T\rangle$.

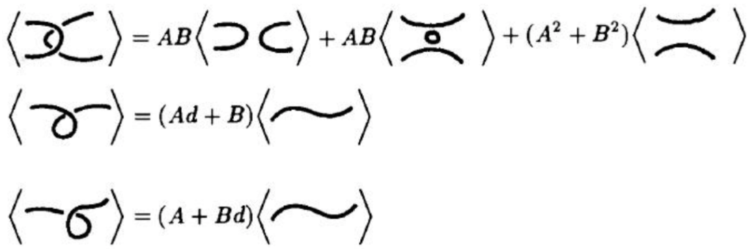


Problem 4. Use the definition to prove that

The notation means that the diagram in the bracket is a part of a larger diagram, and we only change the part shown in the bracket.

Problem 5. (Riedemeister II Invariance).

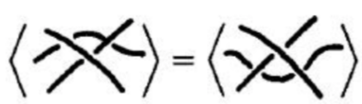
(a) Prove the following identities:



(b) Declare $B = A^{-1}$ and $d = -A^2 - A^{-2}$. With these equalities in place, show that the bracket becomes invariant under Reidemeister II moves.

From now on, we assume $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

Problem 6. (Reidemeister III Invariance). Show that the bracket is invariant under Reidemeister III moves, i.e.



Recall, the **writhe** of an oriented link diagram is the number of positive crossings minus the number of negative crossings.

To make the bracket invariant under Reidemeister I, we need to modify it further. Let K be an oriented link and define the **normalized bracket** of

$$\mathcal{L}_K(A) = (-A^3)^{-w(K)}\langle K\rangle$$

where $w(K)$ is the writhe of K.

Problem 7. Prove that the normalized bracket is invariant under Reidemeister I moves. Conclude that the normalized bracket is an invariant under all Reidemeister moves and hence determines an oriented knot invariant!

Problem 8. (a) Determine the relationship between $\mathcal{L}_K(A)$ and $\mathcal{L}_{K^*}(A)$.
(b) Compute $\langle T^*\rangle$ where T^* is the mirror image of the trefoil.
(c) Compare with $\langle T\rangle$ to conclude that $T \not\sim T^*$.

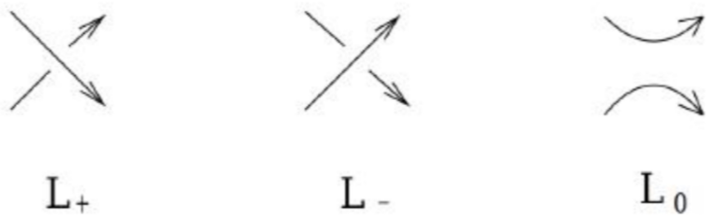
Problem 9. Consider the collection of knots K_n:

(a) Find a recursive formula for $\langle K_n\rangle$.
(b) Show that K_n is not equivalent to K_n^* for any $n > 1$.

2. Jones Polynomial

The Jones polynomial is another (oriented) knot invariant $V_K(t)$ discovered by Vaughan Jones in quite a different context. Its axioms are:

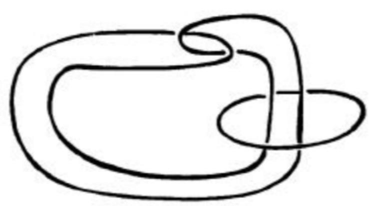
(1) $V_\circ(t) = 1$
(2) If $K \sim K'$ then $V_K(t) = V_{K'}(t)$.
(3) $t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t)$ where L_+, L_-, L_0 are as below:



The existence of V_K is not obvious; the following problem relates it to our earlier construction.

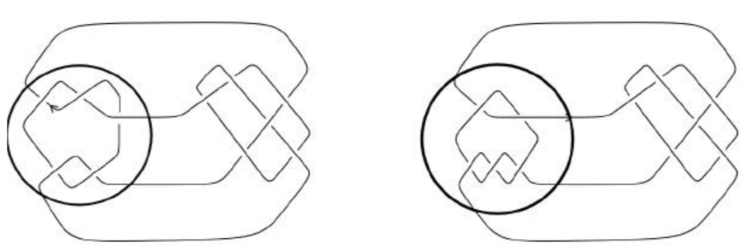
Problem 10. Show that $\mathcal{L}_K(t^{-1/4})$ satisfies (1)-(3). As a result, we can take $V_K(t) = \mathcal{L}_K(t^{-1/4})$.

Problem 11. Let W be the Whitehead link shown below. Prove that $W \not\sim \circ\circ$ by choosing an orientation and computing the Jones polynomial.



Problem 12. Prove using either the Jones polynomial or the Kauffman bracket that the following knot is not equivalent to the unknot.

Problem 13. (Challenge). Unfortunately, Jones' polynomial cannot always detect nonequivalent knots. It turns out knots (a) and (b) are non-equivalent. The transformation shown in the circled region is known as a Conway mutation. Prove that the Conway mutation does not change the Jones polynomial of knots (a) and (b).



(a) Kinoshita-Terasaka knot (b) Conway knot

Problem 14. (Challenge). Choose an orientation of the following two knots and compute their Jones polynomials. Are they equivalent knots? (For the answer see knotplot.com/perko/.)