

## Mike's Birthday Party

You are throwing a party for Mike's birthday. Mike only has 7 friends, including you. They will sit around a table, and each will have a placemat which is either yellow or blue.

1. How many ways are there are to choose a color for each of the 7 placemats?
2. Mike's friends only care about the pattern of colors and not their exact positions. If one placemat arrangement is a rotation of another, they will consider them the same.

For each of the following arrangements, find how many placemat arrangements are "the same":

(These ones have six places, in case Mike doesn't come.)

How is the situation different with 6 placemats instead of 7?

3. If there will be  $p$  people at the table, and  $p$  is a prime number, how many arrangements are the same as a given arrangement if ...

(a) ... all the placemats are the same color?

(b) ... not all the placemats are the same color?

4. How many placemat arrangements are there (up to rotational symmetry) with ...

(a) ... 7 placemats, if each placemat is either blue or yellow?

(Hint: How many monochromatic arrangements are there and how many non-monochromatic arrangements are there?)

(b) ... 7 placemats, if each placemat is either blue, yellow, or red?

(c) ... 5 placemats, if each placemat is one of 25 possible colors?

(d) ...  $p$  placemats, if each placemat is one of  $a$  possible colors?

5. **Fermat's Little Theorem:** *If  $p$  is a prime number and  $a$  is any integer, then  $a^p \equiv a \pmod{p}$ , i.e.  $a^p - a$  is divisible by  $p$ , i.e.  $\frac{a^p - a}{p}$  is a whole number.*

Using the previous problems, give a proof of Fermat's Little Theorem.

## More Examples of Groups

### The integers modulo $n$

$\mathbf{Z}/n\mathbf{Z}$  denotes the integers modulo  $n$ , i.e. the numbers  $0, 1, 2, \dots, n-1$ . There are two natural operations – addition and multiplication mod  $n$ . Unless otherwise specified,  $\mathbf{Z}/n\mathbf{Z}$  refers to additive group, i.e. the set  $\{0, 1, 2, \dots, n\}$  under with addition modulo  $n$  as the operation.

### The dihedral group of order $2n$

$D_{2n}$  denotes the *dihedral group* of order  $2n$  (we write it this way because the order is always an even number) is the group symmetries of a regular  $n$ -gon. There are  $n$  rotations and  $n$  reflections.

### The symmetric group on $n$ elements

$S_n$  denotes the *symmetric group* on  $n$  elements, which is the group of permutations of the numbers  $\{1, 2, \dots, n\}$ .

A permutation of  $\{1, 2, \dots, n\}$  is a way of rearranging these numbers into a different order. One way of writing down a permutation is to make two rows of numbers, with the second row showing where each number ends up:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

## Abelian Groups

We say that a group is *abelian* (named after the mathematician Niels Henrik Abel) if in addition to the normal axioms, the group operation also has the commutative property. When a group is abelian, it is common to write the operation with a  $+$  sign:

$$X + Y = Y + X.$$

## Products of Groups

If  $G$  and  $H$  are groups, the *product group*  $G \times H$  is the group whose elements are ordered pairs  $(g, h)$ , where  $g$  is an element of the group  $G$  and  $h$  is an element of the group  $H$ . We multiply order pairs simply by multiplying in each entry:

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

### Example:

$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$  consists of the ordered pairs  $(0,0)$ ,  $(0,1)$ ,  $(0,2)$ ,  $(1,0)$ ,  $(1,1)$ , and  $(1,2)$ , with the operation of addition mod 2 in the first entry, and addition mod 3 in the second entry, e.g.

$$(0, 1) + (1, 0) = (1, 1)$$

$$(1, 2) + (1, 1) = (0, 0)$$

$$(0, 2) + (1, 2) = (1, 1)$$

(Note that we have chosen to use the symbol '+' for the group operation because the group is abelian, even though we talked about "multiplication" above.)

## Some Groups Were Created More Equal than Others

We've discussed what it means for two groups to be “the same” – the multiplication table of one group looks like the multiplication table for the other, after relabeling the group elements.

1. For each pair of multiplication tables, decide if they are “the same”:

2. (a) Show that  $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$  is “the same as”  $\mathbf{Z}/6\mathbf{Z}$ .

You may want to compare “multiplication” (i.e. addition) tables.

(b) Show that  $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$  is not “the same as”  $\mathbf{Z}/8\mathbf{Z}$ .

3. (a) Show that  $S_3$  is “the same” as  $D_6$ .

(b) Are they “the same” as  $\mathbf{Z}/6\mathbf{Z}$ ?