Hall's Marriage Theorem

Jacob Zhang, Shend Zhjeqi

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1 Hall's Marriage Theorem

Definition 1.1. A finite undirected **graph** $G = (V, E)$ is a finite set V of **vertices**, together with a set E of **edges**, which are formally unordered pairs (v_1, v_2) of elements of V.

Definition 1.2. A graph $G = (V, E)$ is called **bipartite** if we can partition $V = A \cup B$ such that there are no edges $(v_1, v_2) \in E$ with both $v_1, v_2 \in A$ or both $v_1, v_2 \in B$.

Definition 1.3. A **perfect matching** on a bipartite graph $G = (A \cup B, E)$ is a set M of disjoint edges (i.e. no two edges in the set share any vertices) such that every vertex of A is contained in some edge of M.

Definition 1.4. Given a graph $G = (V, E)$ and a subset $S \subseteq V$ of vertices, the **neighborhood** $N(S)$ is the set of vertices connected by an edge to a vertex in S.

Definition 1.5. A bipartite graph $G = (A \cup B, E)$ satisfies **Hall's condition** if for all subsets $S \subseteq A$, $|N(S)| \geq |S|$.

Theorem 1 (Hall's Marriage Theorem)**.** *Let $G = A \cup B$ be a bipartite graph satisfying Hall's condition. Then there exists a perfect matching on G from A to B.*

1.1 Hall's problems

1. A 52-card deck is dealt into 13 rows of 4 cards each. Prove that you can always select one card from each row such that you get exactly one card of each rank.

2. A graph is called **regular** if every vertex has the same degree. Show that every regular bipartite graph admits a perfect matching.

3. A Latin square is an arrangement of the numbers $1, 2, \ldots, n$ in an $n \times n$ square, such that each number appears once in every row and once in every column.

   (a) Show that if we fill in the first k rows of a Latin square, such that each number appears once in every row and at most once in every column, then we can complete the square to $n \times n$ (i.e. there exists an $n \times n$ Latin square with the first k rows being exactly those given).

   (b) Show that if we fill in the numbers $1, 2, \ldots k$ of a Latin square, such that each number from 1 to k appears exactly once in each row and column, then we can fill in the numbers $k + 1, \ldots, n$ to make a complete Latin square.

   (c) (Google Code Jam 2020) For which values of n and m does there exist an $n \times n$ Latin square with diagonal elements summing to m?

4. There are a number of rooks on an $n \times n$ chessboard such that any row or column contains exactly $m \geq 1$ of them. Prove that it is possible to remove some rooks, so that exactly n non-attacking rooks remain.

5. Two 1x1 square pieces of paper are divided into n regions by drawing smooth curves on them. The squares are then placed on top of each other. Show that it is possible to place n pins through the two pieces of paper such that all $2n$ total regions are pierced.