Euler's formula

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Theorem 1: Let G be a (finite) connected planar graph (with no self-intersections) that has V vertices, E edges, and F faces(including the outer face).
Then, $V - E + F = 2$.

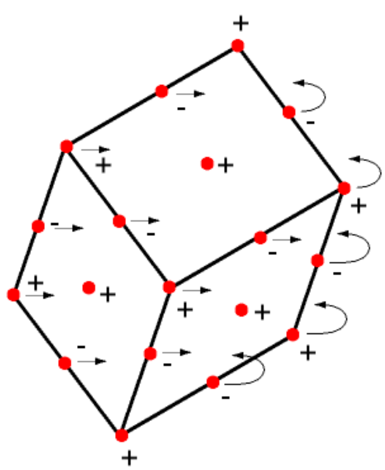
Equivalently,
Theorem 2: Let P be a polyhedra. Say it has V vertices, E edges, and F faces. Then, $V - E + F = 2$.

Question: Why are those theorems equivalent?

1 Proofs of the theorems

Proof 1(of theorem 1): We induct on the number of faces. If G has only one face, the problem is obvious. Otherwise, choose an edge e connecting two different faces of G, and remove it; e appears once in the boundary of each face, and the graph remains connected -any path involving e can be replaced by a path around the other side of one of the two faces. This removal decreases both the number of faces and edges by one, and the result then holds by induction.

Proof 2(of theorem 2): Arrange the polyhedron in space so that no edge is horizontal – in particular, so there is exactly one uppermost vertex U and lowermost vertex L. Put a unit + charge at each vertex, a unit − charge at the center of each edge, and a unit + charge in the middle of each face. We will show that the charges all cancel except for those at L and at U. To do this, we displace all the vertex and edge charges into a neighboring face, and then group together all the charges in each face. The direction of movement is determined by the rule that each charge moves horizontally, counterclockwise as viewed from above.



In this way, each face receives the net charge from an open interval along its boundary. This open interval is decomposed into edges and vertices, which alternate. Since the first and last are edges, there is a surplus of one -; therefore, the total charge in each face is zero. All that is left is +2, for L and for U.

2 Problems

1. Calculate the triples (V, E, F) for the

   a) tetrahedra

   b) octahedron

   c) cube.

2. Triangulate the sphere into two different ways and calculate $V - E + F$.

3. Triangulate a doughnut into two different ways and calculate $V - E + F$.

4. Triangulate mobius strip and find $V - E + F$

5. Let P a polyhedra, let V, E, F be associated to it. Let $f(x) = x^6 + x^5 + x^4 + x^3 + Vx^2 - Ex + F$. Show that f is reducible.

6. A platonic solid has congruent faces (identical in shape and size), regular (all angles equal and all sides equal), polygonal faces with the same number of faces meeting at each vertex. Classify all platonic solids.

7. Three houses in a small village all need to be supplied with water, gas and electricity. Can you make all the connections between the houses and the supplies without having any two cable crossing?

8. The last problem above, but assume that the world is a torus.

3 Discussion

Euler characteristic, nice spaces, and deformations, classification of 2 dimensional surfaces.

What is the euler characteristic of a 2 dimensional surface with g holes (that is compact, connected, orientable)?