# Mean Inequalities Solutions

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1. (a) $(a-b)^2 \geq 0 \implies a^2+b^2-2ab \geq 0 \implies a^2+b^2+2ab \geq 4ab \implies (a+b)^2 \geq 4ab \implies a+b \geq 2\sqrt{ab}$

   (b) $(a-b)^2 \geq 0 \implies a^2+b^2-2ab \geq 0 \implies a^2+b^2 \geq 2ab \implies a^2+b^2+a^2+b^2 \geq a^2+b^2+2ab \implies 2(a^2+b^2) \geq (a+b)^2 \implies \sqrt{2}\sqrt{a^2+b^2} \geq (a+b) \implies \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$

   (c) As in part (a), we have that $(a+b)^2 \geq 4ab$. This implies that $(a+b) \geq 2\sqrt{ab}$, and multiplying both sides by $\frac{\sqrt{ab}}{a+b}$, we get $\sqrt{ab} \geq \frac{2ab}{a+b}$

2. (a) Let $a = x^2, b = y^2$, then this is just the AM-GM inequality.

   (b) Let $a = x, b = \frac{1}{x}$ then this is again AM-GM inequality.

   (c) Let $a = x, b = y$, then HM-AM inequality gives $\frac{xy}{x+y} \leq \frac{x+y}{4}$. Taking reciprocals of both sides (and flipping the inequality), we get the desired result after noticing that $\frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}$.

3. Notice that our proof for each inequality started with $(a-b)^2 \geq 0$. Now if $a = b$, then this is an equality and so all further steps are equalities (since every step after this in our proofs was just manipulation). If $a \neq b$, then this is a strict inequality, so all further steps are strict inequalities. Therefore, we have equality if and only if $a = b$.

4. Let a and b be non-negative numbers. Then we are told that $ab > a+b$. We have that $(a+b)^2 \geq 4ab$, but $4ab > 4(a+b)$. Therefore, $(a+b)^2 > 4(a+b)$. Now if $a+b = 0$, then both a and b are zero and the product is not greater than the sum. Therefore, $a+b \neq 0$, so we can divide by it, giving $a+b > 4$.

5. (a) Using the square of the GM-QM inequality, we have that $xy \leq \frac{x^2+y^2}{2}, xz \leq \frac{x^2+z^2}{2}, yz \leq \frac{z^2+y^2}{2}$. Adding these together, we have

$xy + yz + zx \leq x^2 + y^2 + z^2$. It is worth noting that the square of the inequality does not put restrictions on x,y, and z, so this holds for all values.

(b) We start off by expanding $(ab + bc + ca)^2 = (ab)^2 + (ac)^2 + (bc)^2 + 2abc(a+b+c)$. Then using part (a), with $x = ab, y = ac, z = bc$, then we have that $(ab)^2+(ac)^2+(bc)^2 \geq abc(a+b+c)$, and so combining the two we have $(ab+bc+ca)^2(a+b+c)+2abc(a+b+c) = 3abc(a+b+c)$.

(c) Set $z = 1$ in part (a).

(d) We have that $(a-2b)^2+(a-2c)^2+(a-2d)^2+(a-2e)^2 \geq 0$. Expanding this out, we get $4(a^2 + b^2 + c^2 + d^2 + e^2) - 4a(b + c + d + e) \geq 0$. Adding over and dividing by 4 gives the desired result.

6. Let $a = \frac{81}{x^2}, b = 16x^2$. Then the AM-GM inequality tells us that $a + b \geq 2\sqrt{ab}$, which is just $\frac{81+16x^4}{x^2} \geq 2\sqrt{81 * 16} = 72$. Moreover, this is equality if and only if $a = b$, so $x^4 = \frac{81}{16}$, or $x = \pm\frac{3}{2}$. Therefore the smallest value is 72 and it is acheived at $x = \pm\frac{3}{2}$.

7. (a) We see that this holds for $n = 1$. Now suppose it holds for $n = k$, that is $\sqrt[k]{x_1 x_2 ... x_k} \leq \frac{x_1+x_2+...+x_k}{k}$. Then we have $\sqrt[2k]{x_1...x_k x_{k+1}...x_{2k}} = (\sqrt[k]{x_1...x_k}\sqrt[k]{x_{k+1}...x_{2k}})^{1/2} \leq \frac{\sqrt[k]{x_1...x_k}+\sqrt[k]{x_{k+1}...x_{2k}}}{2}$ by our AM-GM for two variables. But by our inductive hypothesis, we have $\frac{\sqrt[k]{x_1...x_k}+\sqrt[k]{x_{k+1}...x_{2k}}}{2} \leq \frac{\frac{x_1+...x_k}{k}+\frac{x_{k+1}+...+x_{2k}}{k}}{2} = \frac{x_1+...+x_{2k}}{2k}$. Therefore, by induction, we have the statement for all powers of 2.

Now we hope to extend this to all positive integers n. We will show that if it holds for n, then it holds for n-1. Suppose the statement holds n. Then we have $\sqrt[n]{x_1...x_n} \leq \frac{x_1+...+x_n}{n}$.
$\alpha = \frac{x_1+...+x_{n-1}}{n-1} = \frac{\frac{n}{n-1}(x_1+...+x_{n-1})}{n} = \frac{x_1+...x_{n-1}+\frac{1}{n-1}(x_1+...+x_{n-1})}{n} = \frac{x_1+...+x_{n-1}+\alpha}{n} \geq \sqrt[n]{x_1...x_{n-1}\alpha}$. So taking both sides to the nth power, we have $\alpha^n \geq x_1...x_{n-1}\alpha$, or $\alpha^{n-1} \geq x_1...x_{n-1}$. Taking the (n-1)th root of both sides gets the desired inequality.

So we know that if n holds then 2n holds and we know that if n holds then n-1 holds. Using these two properties, we can show it holds for all n. Specifically, we use strong induction. Suppose the statement holds for all $n \leq k$. Then if $k + 1$ is even, then the statement holds for $\frac{k+1}{2}$, and so it holds for $k + 1$. If $k + 1$ is odd, then the statement holds for $\frac{k+2}{2}$ and so it holds for $k+2$ and therefore it holds for $k+1$.

(b) Use part (a) and set $x_1 = \frac{a_1}{a_2}, x_2 = \frac{a_2}{a_3}, ..., x_n = \frac{a_n}{a_1}$. Then we have $\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_n}{a_1} = x_1 + ... + x_n \geq n\sqrt[n]{x_1 x_2 ... x_n} = n\sqrt[n]{1} = n.$

(c) We have by part (a) that $\sqrt[4]{(x^4)(y^4)(1)(1)} \leq \frac{x^4+y^4+1+1}{4}$, so $4xy \leq x^4 + y^4 + 2$. Now if we want equality, then as before we must have $x^4 = y^4 = 1 = 1$, so $x, y = \pm 1$. Taking into account that $xy$ must be positive, the only solutions are $x = y = 1$ and $x = y = -1$.

8. (a) By the AM-QM inequality, we have that $\frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}}$, so plugging in $x + y = 10$, we have that $x^2 + y^2 \geq 50$, with equality if and only if $x = y = 5$. Therefore, the minimum value is 50 when $x = y = 5$.

   To find the maximum, we set $y = 10 - x$ and substitute into $x^2 + y^2$ to get $2x^2 - 20x + 100 = 2(x^2 - 10x + 50)$. This is an upwards facing parabola, so it's maximum will be at the end of the limits so when $x = 0$ or $x = 10$. Therefore, we get a maximum of 100 at $x = 0, y = 10$ or $x = 10, y = 0$.

   Note that we can also find the minimum with our substitution.

   (b) Using the inequalities for n-variables, specifically the AM-GM, we have $\sqrt[20]{x_1 x_2 ... x_{20}} \leq \frac{x_1+x_2+...+x_{20}}{20}$. But since $x_1 x_2 ... x_{20} = (1 - x_1)(1-x_2)...(1-x_{20})$, we also have that $\sqrt[20]{x_1 x_2 ... x_{20}} \leq \frac{(1-x_1)+(1-x_2)+...+(1-x_{20})}{20}$. Adding these two inequalities, we get $2\sqrt[20]{x_1 x_2 ... x_{20}} \leq \frac{20}{20} = 1$. So therefore, $x_1 x_2 ... x_{20} \leq (\frac{1}{2})^2 0$, with equality if $x_i = \frac{1}{2}$ for all i.

9. (a) $HM \leq GM \leq AM \leq QM$

   (b) Using the AM-GM inequality, we have $a + b + c + d \geq 4\sqrt[4]{abcd}$ and $\frac{1}{a} + \frac{1}{b} + frac1c + \frac{1}{d} \geq 4\sqrt[4]{\frac{1}{abcd}}$.
   Then multiply the two, we get $(abcd)(\frac{1}{a} + \frac{1}{b} + frac1c + \frac{1}{d}) \geq 16$.

   (c) Here we use AM-QM. We have that $2 = \frac{a+b+c}{3} \leq \sqrt{\frac{a^2+b^2+c^2}{3}}$. Then we get $a^2 + b^2 + c^2 \geq 12$.

10. Suppose we have blots $b_1, b_2, ..., b_n$. Now if we give our paper coordinates $(x, y)$, then each blot is a connected set in $\mathbb{R}^2$ whose elements are points in $\mathbb{R}^2$. Define the width of blot $b_i$ to be $w_i := max\{x : (x, y) \in b_i\} - min\{x : (x, y) \in b_i\}$. That is, this is the width of the set of vertical lines which intersect the blot. Similarly, we can define the height to be $h_i := max\{y : (x, y) \in b_i\} - min\{y : (x, y) \in b_i\}$, this is the height of the

set of horizontal lines which intersect the blot.

Now since each vertical line intersects at most one blot, we must have that $\sum w_i \leq A$, because otherwise that would imply that the vertical lines for different blots overlap. Similarly, $\sum h_i \leq A$.

Moreover, we have that the area of a blot is bounded by the product of its width and height, $Area(b_i) \leq w_i h_i$.

Since $Area(b_i) \leq 1$, we have that $Area(b_i) \leq \sqrt{Area(b_i)}$ for all i, so $Area(b_i) \leq \sqrt{Area(b_i)} \leq \sqrt{w_i h_i} \leq \frac{w_i + h_i}{2}$, where the last inequality is AM-GM. Summing over all of the blots, this gives $\sum Area(b_i) \leq \sum \frac{w_i + h_i}{2} = \frac{1}{2}(\sum w_i + \sum h_i) \leq \frac{1}{2}(A + A) = A$. But we have that the total area is precisely $\sum Area(b_i)$, so we have that the total area is at most A.