Lesson 6: Bipartite Graphs and Geometry

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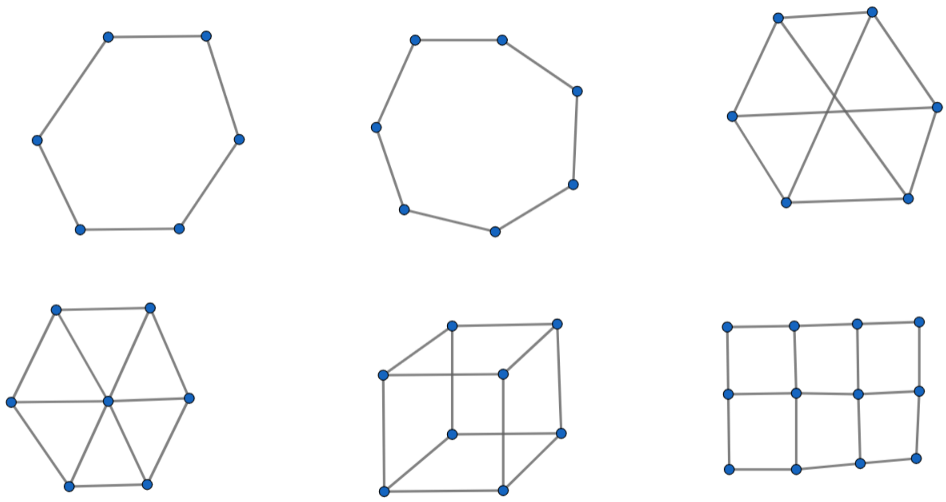
Problem 0.
Are the following six graphs bipartite?



Proof. Yes, No, Yes, No, Yes, Yes □

Problem 1.
Show that a graph is bipartite if and only if it does not have odd cycles. *Hint: parts and special cases of this problem have appeared in the last week's lesson and homework. Look through them to see what exactly you have left to prove.*

Proof. The only direction left to prove is that if a graph has no odd cycles, then it is bipartite. Without going into details, here is the idea – we use the same process we used when coloring the graph without cycles in L5.3. Now we may have cycles, so we may accidentally attempt to recolor an already colored vertex. However, since all the cycles are even, the new color will always be the same as the color already assigned. □

Problem 2.
Show that a graph where every vertex has degree 2 is a collection of disjoint simple cycles.

Proof. Pick any vertex, it has two edges connecting to it. Take one of them, and walk along that edge to a new vertex. That vertex has a second edge, so walk along that. The only way we stop this process is via returning to an already visited vertex. If this is not the vertex we started with, then it would have to have degree 3, contradiction. Thus we eventually return exactly at the starting vertex, which yields a simple cycle. Delete this cycle, and conclude by induction. □

Problem 3.
On a test every student solved exactly 2 problems, and every problem was solved by exactly 2 students.
a) Show that the number of students in the class and the number of problems on the test are the same.

Proof. If we let the students and the problems be vertices and connect a student to a problem if that student solved that problem, we get a bipartite graph and this follows from H5.1. □

b) The teacher wants to make every student present one problem they solved at the board. Show that it is possible to choose the problem each student presents so that every problem on the test gets presented exactly once.

Proof. By problem 2, the graph constructed in part a) is in fact a collection of disjoint cycles. Since the graph is bipartite the cycles are even, and in each cycle the vertices corresponding to students and problems alternate. Then let each student present the next problem clockwise in his cycle, and we are done. □

Problem 4.
An angle bisector of the angle A of an acute triangle ABC intersects its circumcircle at the point D. Show that $BD = CD$.

Proof. The angles ABD and CBD are equal. So the short arcs BD and CD are equal. So the segments are equal. □

Problem 5.
a) Let ABC be an acute triangle. Let AK and BL be altitudes, and call their intersection H. Show that $\angle BLK = \angle HCK$

Proof. Since $\angle HLC + \angle HKC = 90° + 90° = 180°$ we get that $HLCK$ is cyclic. Then $\angle BLK = \angle HCK$ and we are done. □

b) Show that altitudes of an acute triangle intersect at one point.

Proof. Let T be the intersection of the line CH with AB. Since $\angle AKB = \angle ALB = 90°$ we get that $ALKB$ is cyclic, and $\angle BAK = \angle BLK = \angle HCK$. Since $\angle BAK = 90° - \angle ABC$, we get that $\angle TCB + \angle CBT = 90°$. This implies that CT is the altitude, and we are done. □