MATH CIRCLE: SEQUENCES AND SERIES

LAMC OLYMPIAD GROUP

Let us look at the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\ldots$. This sequence seems to be approaching zero, and yet at no point is it actually equal to zero. So how do we formalize this notion?

**Definition.** A **sequence** of real numbers is an assignment $a : \mathbb{N} \rightarrow \mathbb{R}$, i.e. an ordered set of real numbers $a_1, a_2, a_3, \ldots$ (we may also write $\{a_n\}_{n \geq 1}$ or $\{a_n\}$ for this sequence). Such a sequence is said to **converge** to a real number $L$, called the **limit** of the sequence, if and only if for all offsets $\varepsilon > 0$, there is an integer $N$ depending on $\varepsilon$ such that

$$L - \varepsilon < a_n < L + \varepsilon,$$

for all $n > N$.

We write $\lim_{n \to \infty} a_n = L$.

**Problem 1.** Show that if a sequence $\{a_n\}$ converges to both $L$ and $K$, then in fact $L = K$. That is, the limit is unique if it exists.

**Problem 2.** Show that a convergent sequence is bounded. That is, if $\{a_n\}$ converges to some real number $L$, then there exists a constant $A$ such that $-A < a_n < A$ for all $n \geq 1$.

**Problem 3.** (a) Show that if $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $a_n \leq b_n$, then

$$\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n.$$  

(Note that if $a_n < b_n$, we can still only conclude that $\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n$.)

(b) Show that if $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences with $a_n \leq b_n \leq c_n$, such that $\{a_n\}$ and $\{c_n\}$ converge to the same limit, then $\{b_n\}$ also converges to that limit.

**Problem 4.** Show that if $\{a_n\}$ converges to $L$ and $\{b_n\}$ converges to $K$, then $\{a_n + b_n\}$ converges to $L + K$ and $\{a_n \cdot b_n\}$ converges to $L \cdot K$. (It is also true that if $a_n > 0$ converge to $L$ then $a_n^r$ converge to $L^r$ for $r \in \mathbb{R}$.)

**Example.** Let us compute the limit of $a_n = \frac{3n^2 + 5n + 7}{2n^2 - 3n + 11}$.

$$\lim_{n \to \infty} \frac{3n^2 + 5n + 7}{2n^2 - 3n + 11} = \lim_{n \to \infty} \frac{3 + \frac{5}{n} + \frac{7}{n^2}}{2 - \frac{3}{n} + \frac{11}{n^2}} = \frac{3 + 0 + 0}{2 + 0 + 0} = \frac{3}{2}.$$  

**Definition.** The **series** associated to a sequence $\{a_n\}$ is the new sequence $\{s_n\}$ of **partial sums** given by

$$s_n = a_1 + \cdots + a_n.$$  

We say that the series converges to a real number $S$ if $s$ converges $S$. In that case, we write $\sum_{n=1}^{\infty} a_n = S$ and call this limit the **value** of the series. If the series doesn’t converge, we say that it **diverges**.
Problem 5. Show that if the series $\sum_n a_n$ converges, then the sequence $\{a_n\}$ must converge to 0. (Hint: subtract two consecutive partial sums and use Problem 4.)

Problem 6. (a) Show that if $-1 < r < 1$, then $\sum_{n=0}^\infty r^n = \frac{1}{1-r}$.
(b) We toss a fair coin until we hit heads on the $n$th try. What is the probability that $n$ is even?
(c) Interpreting the base-10 number $0.a_1a_2a_3 \ldots$ as $\sum_{n \geq 1} \frac{a_n}{10^n}$, show that $0.999 \ldots = 1$.

Axiom. Any increasing bounded sequence is convergent.
(To show this we would need to formally construct the real numbers; this is one of their defining features. Note that this doesn’t work if we only allow rational numbers.)

Problem 7. Show the same statement for decreasing sequences.

Problem 8. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$. Show that $\lim_{n \to \infty} a_n = 2$.

Problem 9. Given two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $a_n \leq b_n$ and $\sum_n b_n$ converges, show that $\sum_n a_n$ converges. Hint: Use Problem 2 and the Axiom.

Definition. A subsequence of a sequence $\{a_n\}$ is given by selecting only some terms of the sequence. That is, it is given by $\{a_{k_n}\}$ where $\{k_n\}$ is a strictly increasing sequence of indices $k_1 < k_2 < \cdots$.

Problem 10. Show that any subsequence of a convergent sequence converges to the same limit.

Problem 11. Given any two sequences $\{a_n\}$ and $\{b_n\}$ both converging to the same number $L$, show that the new sequence
\[ c_n = \begin{cases} a_{(n+1)/2} & \text{if } n \text{ is odd} \\ b_{n/2} & \text{if } n \text{ is even} \end{cases} \]
given by interlacing $a$ and $b$ also converges to $L$.

Problem 12. (a) Show that $\sum_n \frac{1}{n}$ diverges.
(b) Show that $\sum_{n=1}^\infty \frac{1}{n(n+1)} = 1$.
(c) Show that $\sum_n \frac{1}{n^2}$ and $\sum_n \frac{(-1)^n}{n}$ converge. Hint for the first one: use part (b) and Problem 9; Hint for the second one: look at odd and even partial sums, and use Problem 11.

Problem *13. Suppose that $\sum_n a_n$ and $\sum_n b_n$ are two series with the same terms but in a different order, and assume that all $a_n \geq 0$. Show that if $\sum_n a_n$ converges then $\sum_n b_n$ converges to the same limit.

Problem *14. Show that any bounded sequence has a convergent subsequence.

Problem *15. Let $S$ be a nonempty bounded set of real numbers. Show that there is real number $u$ such that $x \leq u$ for all $x \in S$, and which is minimal with this property. This is called a supremum, denoted $u = \sup S$. 

**Definition.** Let \( I \) be an interval of \( \mathbb{R} \). A function \( f : I \to \mathbb{R} \) is said to be continuous if for all points \( p \in I \) and sequences \( a_n \) converging to \( p \), the sequence \( f(a_n) \) is also convergent and
\[
\lim_{n \to \infty} f(a_n) = f \left( \lim_{n \to \infty} a_n \right). \tag{*}
\]
The slogan is: continuous functions commute with taking limits. An alternative way to phrase this is that
\[
a_n \to L \implies f(a_n) \to f(L),
\]
the slogan here being: continuous functions preserve convergent sequences.

**Examples.** The functions \( x^r, a^x, \sin(x) \) are continuous wherever defined.

**Problem 16.** Show that if \( f, g \) are continuous functions, then \( f + g \) and \( f \cdot g \) are also continuous, wherever defined.

**Problem 17.** Show that if \( f, g : \mathbb{R} \to \mathbb{R} \) are continuous functions, then so is their composite \( g \circ f \).

**Problem *18.** Show that in fact (*) follows from the rest of the definition of continuity.

**Theorem.** Let \( I = [a, b] \) be a closed interval, and \( f : I \to \mathbb{R} \) be a continuous function. Then the image of \( f \) (i.e. set of values attained) is also a closed interval.

**Corollary 1.** The function \( f \) attains both a maximum and a minimum.

**Corollary 2.** (Intermediate value property) If a real number \( z \) is between \( f(a) \) and \( f(b) \), then there exists a point \( c \) in \( [a, b] \) so that \( f(c) = z \).

**Problem 19.** Let \( n \) be a positive integer, and let \( f : \{1, 2, \ldots, n\} \to \mathbb{Z} \) be a function such that \( |f(k) - f(k + 1)| \leq 1 \) whenever \( 1 \leq k < n \). Show that if \( z \) is an integer between \( f(1) \) and \( f(n) \), then there exists an integer \( c \) such that \( f(c) = z \).

**Problem 20.** Show that there are two antipodal points on Earth’s equator with the same altitude. (Assume altitude is a continuous function.)