MATH CIRCLE: SEQUENCES AND SERIES

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Let us look at the numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots$. This sequence seems to be approaching zero, and yet at no point is it actually equal to zero. So how do we formalize this notion?

Definition. A *sequence* of real numbers is an assignment $a : \mathbf{N} \to \mathbf{R}$, i.e. an ordered set of real numbers $a_1, a_2, a_3, \ldots$ (we may also write $\{a_n\}_{n\geq 1}$ or $\{a_n\}$ for this sequence). Such a sequence is said to *converge* to a real number L, called the *limit* of the sequence, if and only if for all offsets $\varepsilon > 0$, there is an integer N depending on ε such that

$$L - \varepsilon < a_n < L + \varepsilon, \text{ for all } n > N.$$

We write $\lim_{n\to\infty} a_n = L$.

Problem 1. Show that if a sequence $\{a_n\}$ converges to both L and K, then in fact $L = K$. That is, the limit is unique if it exists.

Problem 2. Show that a convergent sequence is bounded. That is, if $\{a_n\}$ converges to some real number L, then there exists a constant A such that $-A < a_n < A$ for all $n \geq 1$.

Problem 3. **(a)** Show that if $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $a_n \leq b_n$, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

(Note that if $a_n < b_n$, we can still *only* conclude that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.)
(b) Show that if $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences with $a_n \leq b_n \leq c_n$, such that $\{a_n\}$ and $\{c_n\}$ *converge to the same limit*, then $\{b_n\}$ also converges to that limit.

Problem 4. Show that if $\{a_n\}$ converges to L and $\{b_n\}$ converges to K, then $\{a_n + b_n\}$ converges to $L + K$ and $\{a_n \cdot b_n\}$ converges to $L \cdot K$. (It is also true that if $a_n > 0$ converge to L then ${a_n}^r$ converge to L^r for $r \in \mathbf{R}$.)

Example. Let us compute the limit of $a_n = \frac{3n^2+5n+7}{2n^2-3n+11}$:

$$\lim \frac{3n^2 + 5n + 7}{2n^2 - 3n + 11} = \lim \frac{3 + \frac{5}{n} + \frac{7}{n^2}}{2 - \frac{3}{n} + \frac{11}{n^2}} = \frac{3 + 0 + 0}{2 + 0 + 0} = \frac{3}{2}.$$

Definition. The *series* associated to a sequence $\{a_n\}$ is the new sequence $\{s_n\}$ of *partial sums* given by

$$s_n = a_1 + \cdots + a_n.$$

We say that the series converges to a real number S if s converges S. In that case, we write $\sum_{n=1}^{\infty} a_n = S$ and call this limit the *value* of the series. If the series doesn't converge, we say that it *diverges*.

Problem 5. Show that if the series $\sum_n a_n$ converges, then the sequence $\{a_n\}$ must converge to 0. (*Hint: subtract two consecutive partial sums and use Problem 4.*)

Problem 6. **(a)** Show that if $-1 < r < 1$, then $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.
(b) We toss a fair coin until we hit heads on the nth try. What is the probability that n is even?
(c) Interpreting the base-10 number $0.a_1a_2a_3\ldots$ as $\sum_{n\geq 1} \frac{a_n}{10^n}$, show that $0.999\ldots = 1$.

Axiom. Any increasing bounded sequence is convergent.
(To show this we would need to formally construct the real numbers; this is one of their defining features. Note that this doesn't work if we only allow rational numbers.)

Problem 7. Show the same statement for decreasing sequences.

Problem 8. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$. Show that $\lim_{n\to\infty} a_n = 2$.

Problem 9. Given two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $a_n \leq b_n$ and $\sum_n b_n$ converges, show that $\sum_n a_n$ converges. *Hint: Use Problem 2 and the Axiom.*

Definition. A subsequence of a sequence $\{a_n\}$ is given by selecting only some terms of the sequence. That is, it is given by $\{a_{k_n}\}$ where $\{k_n\}$ is a strictly increasing sequence of indices $k_1 < k_2 < \cdots$.

Problem 10. Show that any subsequence of a convergent sequence converges to the same limit.

Problem 11. Given any two sequences $\{a_n\}$ and $\{b_n\}$ both converging to the same number L, show that the new sequence

$$c_n = \begin{cases} a_{(n+1)/2} & \text{if } n \text{ is odd} \\ b_{n/2} & \text{if } n \text{ is even} \end{cases}$$

given by interlacing a and b also converges to L.

Problem 12. **(a)** Show that $\sum_n \frac{1}{n}$ diverges.
(b) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.
(c) Show that $\sum_n \frac{1}{n^2}$ and $\sum_n \frac{(-1)^n}{n}$ converge. *Hint for the first one: use part (b) and Problem 9; Hint for the second one: look at odd and even partial sums, and use Problem 11.*

Problem *13. Suppose that $\sum_n a_n$ and $\sum_n b_n$ are two series with the same terms but in a different order, and assume that all $a_n \geq 0$. Show that if $\sum_n a_n$ converges then $\sum_n b_n$ converges to the same limit.

Problem *14. Show that any bounded sequence has a convergent subsequence.

Problem *15. Let S be a nonempty bounded set of real numbers. Show that there is real number u such that $x \leq u$ for all $x \in S$, and which is minimal with this property. This is called a supremum, denoted $u = \sup S$.

Definition. Let I be an interval of $\mathbf{R}$. A function $f : I \to \mathbf{R}$ is said to be *continuous* if for all points $p \in I$ and sequences a_n converging to p, the sequence $f(a_n)$ is also convergent and

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right). \qquad (*)$$

The slogan is: continuous functions commute with taking limits. An alternative way to phrase this is that

$$a_n \to L \qquad \Longrightarrow \qquad f(a_n) \to f(L),$$

the slogan here being: continuous functions preserve convergent sequences.

Examples. The functions $x^r, a^x, \sin(x)$ are continuous wherever defined.

Problem 16. Show that if f, g are continuous functions, then $f + g$ and $f \cdot g$ are also continuous, wherever defined.

Problem 17. Show that if $f, g : \mathbf{R} \to \mathbf{R}$ are continuous functions, then so is their composite $g \circ f$.

Problem *18. Show that in fact $(*)$ follows from the rest of the definition of continuity.

Theorem. Let $I = [a, b]$ be a *closed* interval, and $f : I \to \mathbf{R}$ be a continuous function. Then the image of f (i.e. set of values attained) is also a closed interval.

Corollary 1. The function f attains both a maximum and a minimum.

Corollary 2. (Intermediate value property) If a real number z is between $f(a)$ and $f(b)$, then there exists a point c in $[a, b]$ so that $f(c) = z$.

Problem 19. Let n be a positive integer, and let $f : \{1, 2, \ldots, n\} \to \mathbf{Z}$ be a function such that $|f(k) - f(k + 1)| \leq 1$ whenever $1 \leq k < n$. Show that if z is an integer between $f(1)$ and $f(n)$, then there exists an integer c such that $f(c) = z$.

Problem 20. Show that there are two antipodal points on Earth's equator with the same altitude. (Assume altitude is a continuous function.)