INVERSION WITH RESPECT TO A CIRCLE

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1. Introduction: transformations of the plane

Transformation of the plane is a rule which specifies where each of the points of the plane is moved. I.e., $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation if for any point X on the plane a unique image $X' = \varphi(X)$ is specified.

Exercise 1. Recall as many transformations of the plane as you can.

Any geometric shape (e.g., a straight line, a circle, a triangle) is transformed into another geometric shape by such a transformation. Denote by $F' = \varphi(F)$ the image of a geometric shape F under the transformation φ.

Typically (but not always!), a point on the plane is moved somewhere else. Point X is called a *fixed point of* φ if $X' = \varphi(X) = X$, i.e., the point is not moved by the transformation.

Exercise 2.

(1) Let l be a line on the plane. Give an example of a transformation such that each of the points of l is fixed but no other points are fixed.
(2) Give an example of a transformation which has exactly one fixed point.
(3) Give an example of a transformation which has exactly two fixed points.
(4) Give an example of a transformation which has no fixed points.

More generally, a geometric shape F is called *invariant under* φ *if* $F' = \varphi(F) = F$, i.e., if F is transformed into itself.

Exercise 3.

(1) Let l be a line on the plane. Give an example of a transformation φ such that l is invariant under φ, i.e., $\varphi(l) = l$, but none of the points of l are fixed points.
(2) Give an example of a transformation φ and a geometric shape F (which is not a line) such that $\varphi(F) = F$, but none of the points of F is a fixed point of φ.

Exercise 4.

(1) Which of the transformations above preserve distances?
(2) Which of the transformations above preserve angles?

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2. Symmetry and reflection with respect to a line

Two points A and A' on the plane are *symmetric with respect to a line* l if the segment AA' is perpendicular to l, and the point of their intersection $O = l \cap AA'$ is the midpoint of AA'.

You can think of l as a mirror, and of A' as an image of A in the mirror. Such a reflection in the (imaginary) mirror is a transformation of the plane: for each point on the plane there is a unique image. To obtain the image of a point A under such a reflection, draw a line perpendicular to l and going through A. Find the point A' on the perpendicular line so that l is a perpendicular bisector of AA'. Then A' is a reflection of A with respect to l.

Problem 5. *Let A and B be two points on the plane. Find the set of all points which are equidistant from A and B. I.e., find the set of all points X on the plane which satisfy*

$$|AX| = |XB|. \tag{2.1}$$

Note that you need to prove two things:

(1) *All the points in your set satisfy this condition.*
(2) *No other points satisfy this condition.*

The answer is closely related to the reflection with respect to a line.

3. Two problems

Apollonian families. An ancient Greek mathematician Apollonius (260 BC - 190 BC) posed the following problem, which is a natural generalization of Problem 5 above:

Problem 6. *Let A and B be two points on the plane, and let k be a positive number. Find the set of all points X on the plane satisfying*

$$|AX| = k \cdot |XB|. \tag{3.1}$$

(Note that in the case $k = 1$ this is exactly problem 1).

To get an idea of what the solution might look like, consider the following:

Problem 7. *In problem 6*

(1) *Fix $k = 2$. How many points X on the line going through AB satisfy (3.1)? Mark these points. Sketch several other points X (not lying on the line through AB) such that the condition 3.1 is (approximately) satisfied. Can you make a guess as to how the solution of (3.1) looks like for $k = 2$? (make a picture)*
(2) *Look at the same problem for $k = 3$.*

Since the solution of Problem 5 is related to reflection with respect to a line, one might expect that a solution of the generalized Problem 6 is related to another transformation of the plane. The question is, what is this transformation?

A construction problem. Apollonius also proposed the following problem:
Given three circles on the plane, construct a circle tangent to all of them.
Comments:

(1) For ancient Greeks, "construct" means "construct using a ruler and a compass".

(2) Two circles are *tangent* to each other if they have exactly one point in common.

A solution of this construction problem (and of many others!) is a based on learning about the same transformation called Inversion.

4. Inversion (with respect to a circle)

Fix a circle of radius R with center at a point O on the plane. We will define the following transformation I of the plane:

For any point $A \neq O$ on the plane, its image $A' = I(A)$ under the transformation is the point on the plane such that

(1) *A' lies on the line going through the center O of the chosen circle, and A.*
(2) $|AO| \cdot |OA'| = R^2$.

This transformation is called *Inversion with respect to a circle.*

Note that inversion is not defined for the center of the chosen circle. This can be dealt with in the following way. Using stereographic projection, one can add an artificial *point at infinity* O_∞ to the plane, and define

$$I(O) = O_\infty, \quad I(O_\infty) = O.$$

The plane together with point at infinity is called the *extended plane*. Considering the extended plane allows us to view straight lines as generalized circles. Indeed any three (ordinary) points on the plane determine a unique circle (the circumscribed circle of the triangle with these vertices). Any two (ordinary) points (plus point at infinity) determine a straight line (which can be viewed as a generalized circle going through the point at infinity).

From now on, unless noted otherwise, I is the inversion with respect to the circle of radius R with center O.

First, we will establish the following properties of Inversion:

Problem 8. (*Elementary properties of inversion)* Prove the following:

(1) Points on the circle of inversion stay fixed. Points inside of the circle of inversion are moved outside. Points outside of the circle of inversion are moved inside.
(2) If A' is the image of A, then A is the image of A'. (In short, $(A')' = A$).
(3) A straight line going through the center of inversion is invariant under the inversion. (I.e., $I(l) = l$ if $O \in l$).

Properties 1 and 2 allow us to call inversion a type of "symmetry".

A lot of useful properties of inversion is based on the following

Lemma 9. *Let A' and B' be the images of points A and B under inversion with center O with respect to a circle of radius r. Then $\triangle OA'B' \sim \triangle OBA$.*

Problem 10. Prove the lemma above (use the definition of inversion and properties of similar triangles).

Problem 11. Prove further properties of lines and circles under inversion:

(1) A line not going though the center of inversion is transformed into a circle going through the center. (And vice versa).

(2) A circle not going through the center of inversion is transformed into a circle.

Considering a straight line as a circle of infinite radius containing the point at infinity, we can summarize these properties of inversion as follows:

Inversion transforms (generalized) circles into (generalized) circles. The circles going through the center of inversion and the circles containing the point at infinity are transformed into each other.

Next, we will see how inversion changes the angles between lines and circles.

Definition 12. The angle between two intersecting circles is the angle between their tangent lines at the point of intersection.
The angle between a line and a circle is the angle between the line and the tangent to the circle at the point of intersection.

Problem 13. *(Angle-preserving properties)*

Show that the angle between two lines is preserved under inversion. Let l_1 and l_2 be two lines. There are three possible cases:

(1) Both l_1 and l_2 go through O (the center of inversion).
(2) l_1 goes through O, but l_2 does not. (there are two possibilities: $l_1||l_2$ and $l_1 \cap l_2 \neq \emptyset$).
(3) Neither l_1 nor l_2 go through O.

The first case is trivial. Prove the second one (under the assumption that $l_1 \cap l_2 \neq \emptyset$). The rest of the cases can be considered analogously.

More generally, one can show that inversion preserves the angles between any two (generalized) circles.

Problem 14. (*The theorem about the square of the tangent)* Let P be a point outside of a circle T with center O. Let PD be tangent to the circle, and let l be a line going through P that intersects the circle at points A and B. Show that

$$|PD|^2 = |PA| \cdot |PB|.$$

(*Hint:* Let DD_1 be the diameter of the circle going through D. Use the fact that $\angle DAB = \angle DD_1B$ and that $\angle D_1BD = 90^0$ to show that $\angle DAP = \angle PDB$. After that, prove the statement of the theorem.)

Problem solving. Use properties of inversion to solve the following problems.

Problem 15. Let A and B be two points on a circle T. Show that if A and B are transformed into each other under the inversion with respect to a circle S, then the entire circle T is fixed un der this inversion.

Problem 16. Let S and T be two circles which are orthogonal (i.e., the angles between their tangents are equal to $90°$ at their points of intersection). Show that S is invariant under inversion with respect to T and T is invariant with respect to inversion under S. (Hint: use problem 14

Problem 17. *(Change of distances under inversion)* Let A' and B' be the images of A and B under inversion with respect to a circle with radius R and center O. Show that

$$|A'B'| = \frac{R^2}{|OA| \cdot |OB|} \cdot |AB| \tag{4.1}$$

(*Hint:* use Lemma 9).

Problem 18. Let MA and MB be two lines going through a point M and tangent to a circle S at the points A and B. Let D be the center of the chord AB. Show that D and M are the images of each other under the inversion with respect to S. (*Hint:* Find a pair of similar triangles).

Problem 19. (Ptolemy's theorem)

Let $ABCD$ be an inscribed quadrilateral. Show that

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|. \tag{4.2}$$

(*Hint:* use inversion with center A. Then the circumscribed circle S is transformed into a line l; the vertices B, C, D are transformed into points B', C', D' lying on l. After that use the obvious fact that $|B'D'| = |B'C'| + |C'D'|$ together with the formula for change of distance (4.1) to get the Ptolemy's formula (4.2).

Construction problems involving Inversion. The result of problem 18 allows us to construct the image of any point lying outside of the circle of inversion.

Let M be a point outside of the circle of inversion. To find the image of M under the inversion, construct the line through M which is tangent to the circle. Let A be the point of tangency. Construct the line AD perpendicular to MO. Then D is the image of M under the inversion.

Exercise 20. Explain how to construct the line going through a point M lying outside of the given circle and tangent to the circle.

Problem 21. Assume that point M is inside of the circle of inversion. Construct its image under the inversion.

Note that since we can construct the image of any point under inversion, we can also construct the image of any straight line and of any circle. (It's enough to construct the images of any three points lying on the line/circle, and then draw a circle/straight line going through these three points).

Problem 22. Given a line l and two points, A and B, construct a circle S going through A, B and tangent to l. (*Hint:* Assume that such a circle T is constructed. Perform an inversion with center A. What are the images of l and S under this inversion? Use this inversion to solve the original problem).

Problem 23. Use inversion to solve the Apollonius problem 6 in the cases that at least two of the circles are tangent to each other:

Let S, T, Q be three circles on the plane such that S and T are tangent to each other. Let O be the point of tangency. Construct a circle tangent to S, T, Q. (*Hint:* an inversion with center O transforms the circles S and T into a pair of parallel lines S' and T'. The third circle, Q, is transformed into a circle or a straight line Q'. Construct the circle tangent to S', T', Q'. Its image under the same inversion is the required circle).

One can show that several other cases can be reduced to this one.

Let's use inversion to solve the problem posed at the beginning of the handout: find the set of all points X such that $|AX| = k \cdot |BX|$, where A and B are given fixed points on the plane, and k is a given fixed number.

Problem 24. Let A and B be two fixed points. Consider the set of all points X such that $|AX| = k \cdot |BX|$, where k is a fixed positive number. Let X' be the image of X under inversion with respect to a circle with center A. Let B' be the image of B with respect to the same inversion. Describe the set of all points X'. (*Hint:* assume for simplicity that the circle of inversion has unit radius. Compute the distance $|BX'|$.