# INFINITE DECIMALS AND GEOMETRIC SERIES

MATH CIRCLE (BEGINNERS) 10/23/2011

(1) Let's think about the finite decimal $0.375$. We can write it as a sum of fractions with denominators that are powers of 10 like this:

$$0.375 = \frac{3}{10} + \frac{7}{100} + \frac{5}{1000}.$$

(a) Consider the repeating decimal

$$0.\overline{4} = 0.44444444\ldots$$

You can write it as an *infinite* sum of fractions whose denominators are powers of 10. What are the first few summands in this infinite summation? (I've written the fraction bars for you!)

$$0.\overline{4} = \frac{\quad}{\quad} + \frac{\quad}{\quad} + \frac{\quad}{\quad} + \frac{\quad}{\quad} + \ldots$$

(b) What is the value of the infinite sum of fractions you just wrote?

(2) Write the infinite sum

$$\frac{15}{100} + \frac{15}{10000} + \frac{15}{1000000} + \frac{15}{100000000} + \frac{15}{10000000000} + \ldots$$

as a repeating decimal.

(a) Express the value of the infinite sum as a fraction (ratio of integers). (Hint: It's the same as the value of the repeating decimal, which we learned how to compute in a previous handout...)

(3) What are the next two terms that are missing in the following infinite sum?

$$\frac{35}{50} + \frac{35}{5000} + \frac{35}{500000} + \rule{4cm}{0.4pt} + \rule{6cm}{0.4pt} + \dots$$

- Express the infinite sum as a fraction. (Hint: Can you turn the denominators into powers of 10? Then what?)

(4) Let's try to evaluate an infinite sum using a different technique. We'll use a geometric picture to help us. The infinite sum is:

$$\begin{aligned} x &= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \dots \\ &= \frac{1}{4} + \frac{1}{4 \cdot 4} + \frac{1}{4 \cdot 4 \cdot 4} + \frac{1}{4 \cdot 4 \cdot 4 \cdot 4} + \frac{1}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} + \dots \end{aligned}$$

(a) Use a blank page for your drawing. Try to draw a good picture; if it's sloppy it will be harder to see what's going on!

(b) Draw a LARGE square and then draw a vertical and horizontal line to divide it into four smaller, equal squares. (Like a window.)

(c) Lightly shade in the lower left small square.

(d) In the upper right small square, again draw a short horizontal and vertical line to divide it into four even smaller squares—let's call these small-small squares.

(e) Lightly shade the lower left small-small square inside the upper right small square.

(f) Repeat the process in the upper right small-small square: Divide it into 4 small-small-small squares, and shade in the lower left of these.

(g) Repeat the process in the upper right small-small-small square, and keep repeating it until everything's so small that you can't even draw it anymore (but we'll pretend that it just keeps going forever!).

(h) If the area of the large square you drew is 1 square unit, what is the area of the smaller shaded squares? Of course they're not all the same—but write the area next to the shaded square for the largest 3-4 shaded squares.

(i) Write an infinite sum that represents the total area of all the shaded squares in your picture:

**Shaded Area** $= \qquad + \qquad + \qquad + \qquad + \ \dots$

(j) The top left, bottom left, and bottom right small squares make up an L shape in your drawing. Outline this L shape in bold.

(k) Now outline the L shape made up by the top left, bottom left, and bottom right small-small squares. And so on until things get too small to work with.

(l) For each outlined region, what fraction of that region's area is shaded?

fraction of each L shape region that's shaded $=$

(m) Based on your answer in the previous step, what fraction of the LARGE square's area is shaded?

fraction of large square that's shaded $=$

(5) On a new blank page, try drawing another geometric picture that will help you evaluate the infinite sum

$$\begin{aligned} x &= \frac{5}{9} + \frac{5}{81} + \frac{5}{729} + \frac{5}{6561} + \frac{5}{59049} + \dots \\ &= \frac{5}{9} + \frac{5}{9\cdot 9} + \frac{5}{9\cdot 9\cdot 9} + \frac{5}{9\cdot 9\cdot 9\cdot 9} + \frac{5}{9\cdot 9\cdot 9\cdot 9\cdot 9} + \dots \end{aligned}$$

(a) Hint: Start by dividing the square into 9 equal small squares. Then divide the top right square into 9 small-small squares, and then the top right of THAT into 9 small-small-small squares, and repeat until it's too small to work with. Then shade 5 of the 9 small squares, then 5 of the 9 small-small squares, etc. Then draw a new L-shape around the 8 small squares except the top right one. Then what?

(6) The square-drawing way of evaluating an infinite sum was kind of cool, but it's not clear how we can use it on other sums we might want to know the value of... So let's try something different again.

(a) Suppose we want to know the value as a fraction, of the infinite sum:

$$S = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \underline{\hspace{2cm}} + \frac{64}{729} + \dots$$

(Fill in the missing term!)

(b) What happens if we multiply both sides of that equation by $\frac{3}{2}$? (When we multiply an infinite sum by a number, it works just like a finite sum–we can multiply each summand by the number individually.)

$$\frac{3}{2} \cdot S = \frac{3}{2} \cdot \frac{2}{3} + \frac{3}{2} \cdot \frac{4}{9} + \frac{3}{2} \cdot \underline{\hspace{1.5cm}} + \frac{3}{2} \cdot \underline{\hspace{1.5cm}} + \frac{3}{2} \cdot \underline{\hspace{1.5cm}} + \dots$$

If we multiply the fractions and simplify them, we get...

$$\frac{3}{2} \cdot S = 1 + \frac{2}{3} + \underline{\hspace{1.5cm}} + \underline{\hspace{1.5cm}} + \underline{\hspace{1.5cm}} + \dots$$

(c) Look at the right hand side. If you ignore the first "1" and just look at the summands after that, does it look familiar? Can you replace it with a letter??

(d) Once you replace the infinitely many terms with that letter, solve the resulting equation to find the value of $S$:

(7) Now for each of the following infinite sums, fill in the missing term and find the value of the sum using the same technique:

(a)

$$S = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \underline{\qquad\qquad} + \frac{1}{7776} + \dots$$

- Hint: The number to multiply both sides by is $\frac{6}{1} = 6$.

(b)

$$S = \frac{3}{5} + \frac{9}{25} + \underline{\qquad\qquad} + \frac{81}{625} + \frac{243}{3125} + \dots$$

(c)

$$S = \underline{\qquad\qquad} + 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256} + \dots$$

(8) What if you try to apply the multiply-by-a-number technique to the infinite sum

$$S = 1 + 2 + 4 + 8 + 16 + 32 + \dots$$

(a) What should you multiply by?

(b) Can you solve for $S$?

(c) Does this make sense?