Lesson 5: Bipartite graphs and geometry

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Remark 1.
In the beginning of this class, it is necessary to define a bipartite graph.

Problem 1.
Is it possible to walk around the 7×7 chess board with a knight visiting every square exactly once and finishing back at the starting square?

Proof. No. We have to make 49 jumps, and with every jump we switch color under the regular chess coloring. But then we cannot end up at the start since we end up at a color different from the one we started with. This can be interpreted as a bipartite graph where we put all white vertices in one part and black ones in the other ones. Then this graph is indeed bipartite with the edges being the knight moves, so we cannot end up in the starting part after 49 moves. □

Problem 2.
a) A bipartite graph has b white and r black vertices. What is the maximum possible number of edges in this graph?

b) What is the maximum possible number of edges in a bipartite graph with $2n$ vertices?

c) What about $2n + 1$ vertices?

Proof.
a) Since being a bipartite graph forbids us to have edge inter white and black vertices, the maximal number of edges is achieved by connecting every black vertex with every white vertex. The total number of edges is then $b \cdot r$.

b) Suppose there are $n + x$ black vertices and $n - x$ white vertices. Connecting every white vertex with every black vertex, we have $(n + x)(n - x) = n^2 - x^2$ edges. Since $x^2 \geq 0$, this number is maximized when $x = 0$, the total number of edges is then n^2.

c) Suppose there are $n+x+1$ vertices of one color and $n-x$ vertices for the other($0 \geq x \geq n$). Connecting every white vertex with every black vertex, we have $(n+x+1)(n-x) = n^2 + n - x^2 - x$ edges. Since $x^2 + x \geq 0$, this number is maximized when $x = 0$, the total number of edges is then $n^2 + n$. □

Problem 3.
Show that if a graph has no cycles, then it is bipartite.

Proof. We have to color the vertices in two colors so that no edge connects two vertices of the same color. Pick any vertex and color it white. Then color all its neighbors black, all their neighbors white, and so on. You will never attempt to color any vertex twice since there are no cycles. Do this in every connected component, and we are done. □

Problem 4.
In a quadrilateral $ABCD$ angles ABC and ADC are right. Also, $\angle ABD = 40°$. Find $\angle CAD$.

Answer: $50°$

Proof. The quadrilateral $ABCD$ is inscribed. So $\angle ACD = \angle ABD = 40°$ and now we know all the angles in the triangle ACD.

□

Problem 5.
Given two circles with external tangency, prove that the common tangent passing through the tangency point, bisects the segments of external common tangents bounded by the tangency points.

Proof. Let the point of external tangency of the two circles be A, and let the common external tangent to the two circle touch them at points B and C. Let P be the intersection of the tangent through A and BC. Then $PB = PA$ since they are tangents to a given circle from P, and similarly $PC = PA$. Then $PB = PC$, as desired. □

Problem 6.
To two circles tangent externally at a point A, a common external tangent BC is drawn (where B and C are the tangency points). Prove that the angle BAC is right.

Hint: Draw through A a common tangent, define D and examine the triangles ABD and ADC.

Proof. As we showed in the previous problem, $PB = PA = PC$. Then A, B, C lie on the circle with center P and diameter BC, which implies $\angle BAC = 90°$. □

Problem 7.
Let $ABCD$ be a cyclic quadrilateral with $AD = CD$. Let T be the intersection of lines AD and BC. Given that $AB = CT$, show that $\angle DBT = \angle DTB$.

Proof. Since $ABCD$ is cyclic, $\angle ABD = \angle ACD$ and $\angle ADB = \angle ACB$. $\angle DCT = 180° - \angle ACD - \angle ACB$. In triangle ADB, $\angle DAB = 180° - \angle ADB - \angle ABD$. So $\angle DCT = \angle DAB$. We also have $AD = DC, AB = CT$. Therefore $\triangle DAB \cong \triangle DCT$. So $DB = DT$ and thus $\angle DBT = \angle DTB$. □