Let us look at the numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots$. This sequence seems to be approaching zero, and yet at no point is it actually equal to zero. So how do we formalize this notion?

**Definition.** A sequence of real numbers is an assignment $a : \mathbb{N} \to \mathbb{R}$, i.e. an ordered set of real numbers $a_1, a_2, a_3, \ldots$ (we may also write $\{a_n\}_{n \geq 1}$ or $\{a_n\}$ for this sequence). Such a sequence is said to converge to a real number $L$, called the limit of the sequence, if and only if for all offsets $\varepsilon > 0$, there is an integer $N$ depending on $\varepsilon$ such that

$$L - \varepsilon < a_n < L + \varepsilon, \text{ for all } n > N.$$ 

We write $\lim_{n \to \infty} a_n = L$.

**Problem 1.** Show that if a sequence $\{a_n\}$ converges to both $L$ and $K$, then in fact $L = K$. That is, the limit is unique if it exists.

**Problem 2.** Show that a convergent sequence is bounded. That is, if $\{a_n\}$ converges to some real number $L$, then there exists a constant $M$ such that $-M < a_n < M$ for all $n \geq 1$.

**Problem 3.** (a) Show that if $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $a_n \leq b_n$, then

$$\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n.$$ 

(Note that if $a_n < b_n$, we can still only conclude that $\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n$.)

(b) Show that if $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences with $a_n \leq b_n \leq c_n$, such that $\{a_n\}$ and $\{c_n\}$ converge to the same limit, then $\{b_n\}$ also converges to that limit.

**Problem 4.** Show that if $\{a_n\}$ converges to $L$ and $\{b_n\}$ converges to $K$, then $\{a_n + b_n\}$ converges to $L + K$ and $\{a_n \cdot b_n\}$ converges to $L \cdot K$. (It is also true that if $a_n > 0$ converge to $L$ then $a_n^r$ converge to $L^r$ for $r \in \mathbb{R}$.)

**Definition.** The series associated to a sequence $\{a_n\}$ is the new sequence $\{s_n\}$ of partial sums given by

$$s_n = a_1 + \cdots + a_n.$$ 

We say that the series converges to a real number $S$ if $s$ converges $S$. In that case, we write $\sum_{n=1}^{\infty} a_n = S$ and call this limit the value of the series. If the series doesn’t converge, we say that it diverges.

**Problem 5.** Show that if the series $\sum_n a_n$ converges, then the sequence $\{a_n\}$ must converge to 0. (Hint: subtract two consecutive partial sums and use Problem 4.)

**Problem 6.** (a) Show that if $-1 < r < 1$, then $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

(b) We toss a fair coin until we hit heads on the $n$th try. What is the probability that $n$ is even?

(c) Interpreting the base-10 number $0.a_1a_2a_3 \ldots$ as $\sum_{n \geq 1} \frac{a_n}{10^n}$, show that $0.999 \ldots = 1$. 

Axiom. Any increasing bounded sequence is convergent.
(To show this we would need to formally construct the real numbers; this is one of their defining features. Note that this doesn’t work if we only allow rational numbers.)

**Problem 7.** Show the same statement for decreasing sequences.

**Problem 8.** Let \( \{a_n\} \) be the sequence defined recursively by \( a_1 = 1 \) and \( a_{n+1} = \sqrt{2 + a_n} \). Show that \( \lim_{n \to \infty} a_n = 2 \).

**Problem 9.** Given two sequences \( \{a_n\} \) and \( \{b_n\} \) of non-negative integers such that \( a_n \leq b_n \) and \( \sum b_n \) converges, show that \( \sum a_n \) converges.

**Definition.** A subsequence of a sequence \( \{a_n\} \) is given by selecting only some terms of the sequence. That is, it is given by \( \{a_{k_n}\} \) where \( \{k_n\} \) is a strictly increasing sequence of indices \( k_1 < k_2 < \cdots \).

**Problem 10.** Show that any subsequence of a convergent sequence converges to the same limit.

**Problem 11.** Given any two sequences \( \{a_n\} \) and \( \{b_n\} \) both converging to the same number \( L \), show that the new sequence

\[
c_n = \begin{cases} 
  a_{(n+1)/2} & \text{if } n \text{ is odd} \\
  b_{n/2} & \text{if } n \text{ is even}
\end{cases}
\]

given by interlacing \( a \) and \( b \) also converges to \( L \).

**Problem 12.** (a) Show that \( \sum_n \frac{1}{n} \) diverges.

(b) Show that \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \).

(c) Show that \( \sum_n \frac{(-1)^n}{n} \) and \( \sum_n \frac{1}{n^2} \) converge.

**Problem *13.** Suppose that \( \sum_n a_n \) and \( \sum_n b_n \) are two series with the same terms but in a different order, and assume that all \( a_n \geq 0 \). Show that if \( \sum_n a_n \) converges then \( \sum_n b_n \) converges to the same limit.

**Problem *14.** Show that any bounded sequence has a convergent subsequence.

**Problem *15.** Let \( S \) be a nonempty bounded set of real numbers. Show that there is real number \( u \) such that \( x \leq u \) for all \( x \in S \), and which is minimal with this property. This is called a supremum, denoted \( u = \sup S \).