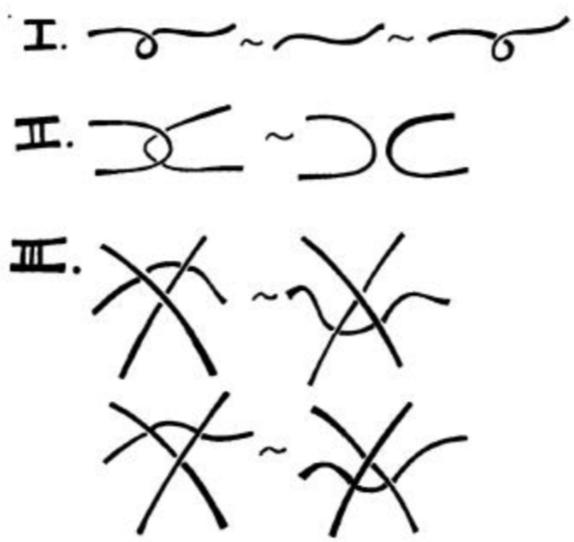
QUANDLES AND KNOT INVARIANTS I

LOS ANGELES MATH CIRCLE
ADVANCED 2
MAY 10, 2020

1. Introduction

A **knot** is a circle "embedded" in three-dimensional space. A **link** is a collection of several knots. We say two knots (or two links) are equivalent, denoted $\sim$, if we can smoothly deform one into the other.



We consider a **diagram** of a knot by projecting it onto a plane and indicating under/over at each crossing. Reidemeister proved that two knots are equivalent if and only if their corresponding diagrams can be transformed into each other using a sequence of the three **Reidemeister moves** shown above.

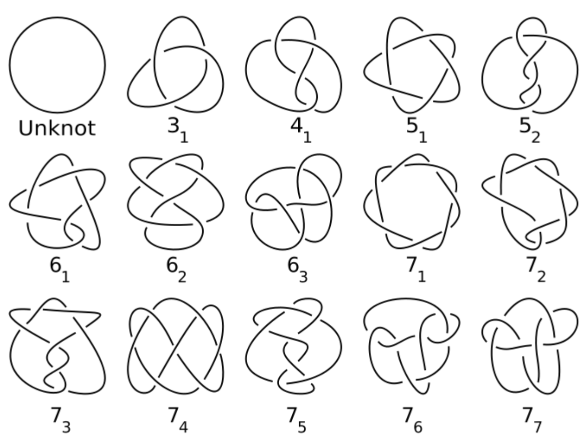
Problem 1. The standard circle circle is called the unknot. Is it true that the circle is equivalent to the following knots? If so, draw a sequence of Reidemeister moves transforming each to the unknot.

(a)

(b)

Problem 2. Let E be the figure-eight knot and consider its mirror image E^* (reverse all the crossings). Draw a sequence of Reidemeister moves to show that $E \sim E^*$.

Problem 3. Let K be any knot and K^* its mirror image. Is it always true that would be $K \sim K^*$? Below are some famous examples of knots to help. Justify your answer.



Problem 4. (Challenge) Give a sequence of Reidemeister moves to show this knot is equivalent to the unknot.

We can give any link an **orientation** by assigning a direction to each link component. We draw this orientation with an arrow. We also record whether each crossing is “positive” or “negative:”

Problem 5. The **writhe** of an oriented link diagram is the number of positive crossings minus the number of negative crossings. Show that the writhe of a link diagram is invariant under Reidemeister moves II and III, but not I.

2. Quandles

Definition 1. A *quandle* is a set X with two binary operations, Q and N (which we pronounce quandle and unquandle), satisfying the following axioms:

(1) The quandle operation preserves identity: For any $x \in X$, $xQx = x$.
(2) The quandle and unquandle operations undo each other: For any $x, y \in X$, $(xQy)Nx = y = xQ(yNx)$.
(3) The quandle operations distribute over itself: For any $x, y, z \in X$, $xQ(yQz) = (xQy)Q(xQz)$ and $(xNy)Nz = (xNz)N(yNz)$.

A set with operations satisfying the latter two axioms is called a "Rack," while a set with an operation satisfying the last operation is called a "Shelf" - but these names are mostly historical oddities, as we'll only be considering quandles today.

Problem 6. Let X be any set and define $xQy = y$, and $xNy = x$ for any elements x, y of X. Verify Q and N satisfy the axioms. Note we have $(xQy)Nx = y$ and $xQ(yNx) = y$, yet we don't have $xN(yQx) = y$ nor do we have $(xNy)Qx = y$. (Only QN works!)

Problem 7. Prove that for any quandle (X, Q, N) and $y \in X$, $y \in X$, $yNy = y$. (Hint: Evaluate $(yQy)Nx$ in two ways.)

Problem 8. Let (X, Q, N) be a quandle. Show that if a and b are two distinct elements of X, aQb cannot be a. So, if X has more than one element, we cannot switch the roles of Q and N in the previous example.

Problem 9. (Challenge) Let $\mathrm{GL}_2(\mathbb{R})$ be the set of 2 by 2 invertible matrices with real entries. For matrices A, B, define $AQB = ABA^{-1}$ and $ANB = B^{-1}AB$. Show Q and N satisfy the above axioms. The two matrices get tangled in a more complicated way than Problem 6, and NA untangles AQ.

Problem 10. (a) Show that if $aQb = aQc$, then $b = c$. (Note that this is not true of $bQa = cQa$.)
(b) Show that if X is associative, then $aQb = b$ for all $a, b \in X$.

Problem 11. Prove that for any $a, b \in X$, there is a unique $x \in X$ such that $aQx = b$.

In light of Problem 11, we can replace the second quandle axiom with the uniqueness property. A quandle X is a set with operation Q such that:

(1) The quandle operation preserves identity: For any $x \in X$, $xQx = x$.
(2) For any $a, b \in X$, there is a unique $x \in X$ such that $aQx = b$.
(3) The quandle operation distributes over itself: For any $x, y, z \in X$, $xQ(yQz) = (xQy)Q(xQz)$.

Problem 12. Two quandles are considered the same if one can be obtained from the other by renaming the elements.

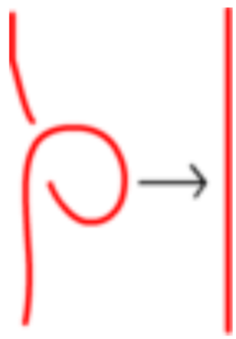
(a) Find all the different quandles of with one element.
(b) Find all the different quandles of with two elements.
(c) (Challenge) Find all the different quandles of with three elements.

Problem 13. Prove that the following are examples of quandles:

(a) The set of non-negative integers less than n, $\{0, 1, 2, \ldots, n-1\}$ with the operation $aQb = 2a - b \mod n$.
(b) The set of points in the plane, with aQb being the point on the opposite side of a from b, the same distance away.
(c) The set of points on the sphere, with aQb being the point that b is sent to when the sphere rotates $180°$ around a.

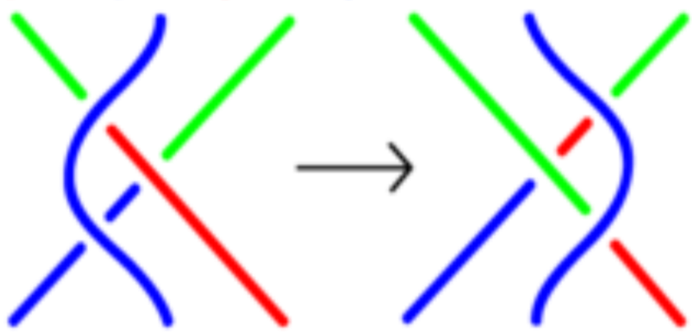
3. Knot Invariants

We have seen a bit of knot theory before, but here's a reminder: a knot is a continuous closed curve in 3 dimensional space that is not allowed to intersect itself. In order to draw knots in two dimensions, we project them onto the plane, with the requirement that intersections only have two arcs hitting at a point, and always indicating which is on top. While two different pictures might represent the same knot, by applying or reversing the Reidemeister moves:



Reidemeister Move 1

Reidemeister Move 2



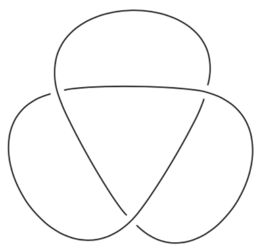
Reidemeister Move 3

we can get between any two depictions of the same knot.

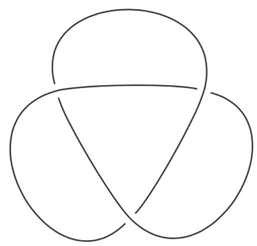
Definition 2. A *knot invariant* is something (like a number, or perhaps something more complicated) associated to a knot diagram, that does not change when any of the Reidemeister moves are applied.

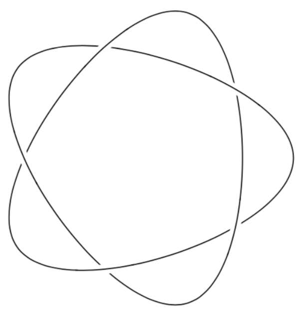
Definition 3. A knot diagram is said to be *tricolorable* if you can assign one of three colors to each unbroken curve in the diagram so that each crossing has either 1 or 3, but never 2, distinct colors, and not all curves are given the same color.

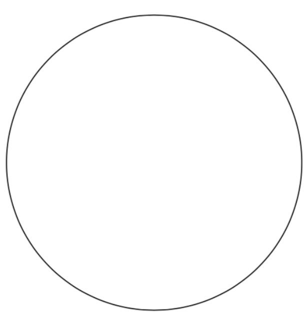
Problem 14. Which of these knots are tricolorable?



.







Problem 15. Prove that tricolorability is an invariant — check that if a knot is tricolored, and you apply or reverse any of the Reidemeister moves, it can remain tricolored.

4. The Knot Quandle

Given any knot, it turns out that you can build a quandle!

Definition 4. The *knot quandle* of a given knot diagram is constructed by labeling each of the arcs in the knot diagram, making sure to label on the same side of the curve as you follow it around. These labels generate the quandle; it is subject to relations obtained at each of the crossings. Assuming the top strand is labeled b and that label is on the righthand side of the strand (rotate the picture if necessary), and that the label on the left strand is a and the right strand is c: if the labels a and c are below their strands, we get the relation $bQa = c$, while if the labels a and c are above their strands, we get the relation $a = cNb$.

Problem 16. Draw what the above relations mean at a crossing.

Problem 17. Determine the knot quandle for each of the knots from Problem 5.

Problem 18. Prove that the knot quandle is a knot invariant.