Lesson 4: Induction in Arithmetic

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Problem 1.
Show that $n^5 - n$ is divisible by 5 for any positive integer n.

Proof. Base $n = 1$ is obvious, for the step notice that

$$(n+1)^5 - n - 1 - (n^5 - n) = 5n^4 + 10n^3 + 10n^2 + 5n = 5(n^4 + 2n^3 + 2n^2 + n)$$

□

Problem 2.
Let x be such a number that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is also integer for $n = 2, 3, \ldots$.

Proof. Strong induction. For $n = 0$ we have 2 is integer. For $n = 1$ it is given. For bigger n

$$\left(x^{n+1} + \frac{1}{x^{n+1}}\right) = \left(x + \frac{1}{x}\right)\left(x^n + \frac{1}{x^n}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)$$

and RHS is integer by the induction hypothesis. □

Problem 3.
a) Show that $n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2$ is divisible by 5 for any positive integer n.

b) Let m be a positive integer not divisible by 2 or 3. Show that $n^2 + (n+1)^2 + \ldots + (n+m-1)^2$ is divisible by m for any positive integer n. Hint: remember the formula

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

which was shown on the board during our very first induction class.

Proof. For the base $n = 1$ we use the sum of squares identity, noting that if $m(m+1)(2m+1)/6$ is an integer and m is not divisible by 2 or 3, then $m(m+1)(2m+1)/6$ is divisible by m. For the step we simply notice that the difference between the expressions for $n+1$ and for n is $(n+m)^2 - n^2 = m(2n+m)$. □

Problem 4.
Show that $3^{2n+2} + 8n - 9$ is divisible by 16 for any positive integer n.

Proof. Base is easy. For the step we have

$$3^{2(n+1)+2} + 8(n+1) - 9 = 9(3^{2n+2} + 8n - 9) - 64n + 80$$

from where the step easily follows. The trick, as usual, is to express the expression for $n+1$ using the expression for n. □

Problem 5.
Show that in a quadrilateral $ABCD$ we have $\angle ABD = \angle ACD$ if and only if points A, B, C, D lie on the same circle. Such a quadrilateral is called *cyclic* or *inscribed*.

Proof. If the points already lie on a single circle, then $\angle ABD$ and $\angle ACD$ are both half the arc AD and thus are equal. Now suppose we have the angle equality, and point C does not lie on the circumcircle of $\triangle ABD$. Then it is either outside or inside that circumcircle. Suppose it is inside, the outside case is similar. Then draw DC to the intersection with the circle, call it T. Then $\angle ABD = \angle ATD$ by the forward direction, which implies that $\angle ATD = \angle ACD$. But $\angle ATD = \angle ACD = \angle ATD + \angle TAC$, which is a contradiction. □

Problem 6.
Show that if in a quadrilateral $ABCD$ we have $\angle ABC + \angle ADC = 180°$, then it is cyclic. This provides a converse to the problem 4 from last week, and gives us another characterization of a cyclic quadrilateral.

Proof. The proof is very similar to the proof to the previous problem. □

Problem 7.
Two circles intersect at the points A and B. Through A, a secant is drawn intersecting the circles at the points C and D. Through B the secant is drawn intersecting the circles at the points E and F. Prove that the quadrilateral $CDFE$ is a trapezoid.

Proof. $\angle CEB = 180° - \angle CAB = \angle DAB = 180° - \angle DFB$, so CE is parallel to DF. □

Problem 8.
a) Show that for a prime $p > 3$ we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(p-1)^2} \equiv 0 \pmod p$$

Proof. Since all residues mod p other than ± 1 break up as pairs of inverses, summing the inverses squared is the same as summing the original residues' squares. Then this is equivalent to showing that

$$1^2 + 2^2 + \ldots + (p-1)^2 \equiv 0 \pmod p$$

which is evident from the sum of squares formula. □

b) Show that for a prime $p > 3$ we have

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1} \equiv 0 \pmod{p^2}$$

Proof. After grouping $1/k$ and $1/(p-k)$ together for $k \leq (p-1)/2$ we can write

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1} = pQ$$

where

$$Q = \frac{1}{1(p-1)} + \frac{1}{2(p-2)} + \ldots + \frac{1}{\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)}$$

Thus it is enough to show that $Q \equiv 0 \pmod{p}$. But

$$Q \equiv -\left(\frac{1}{1^2} + \ldots + \frac{1}{\left(\frac{p-1}{2}\right)}\right) \equiv \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(p-1)^2}\right) \equiv 0 \pmod{p}$$

by part a). □