Lesson 4: Induction in Arithmetic

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Problem 1.
Show that $n^5 - n$ is divisible by 5 for any positive integer n.

Problem 2.
Let x be such a number that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is also integer for $n = 2, 3, \ldots$.

Problem 3.
a) Show that $n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2$ is divisible by 5 for any positive integer n.

b) Let m be a positive integer not divisible by 2 or 3. Show that $n^2 + (n+1)^2 + \ldots + (n+m-1)^2$ is divisible by m for any positive integer n. Hint: remember the formula

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Problem 4.
Show that $3^{2n+2} + 8n - 9$ is divisible by 16 for any positive integer n.

Problem 5.
Show that in a quadrilateral $ABCD$ we have $\angle ABD = \angle ACD$ if and only if points A, B, C, D lie on the same circle. Such a quadrilateral is called *cyclic* or *inscribed*.

Problem 6.
Show that if in a quadrilateral $ABCD$ we have $\angle ABC + \angle ADC = 180°$, then it is cyclic. This provides a converse to the problem 4 from last week, and gives us another characterization of a cyclic quadrilateral.

Problem 7.
Two circles intersect at the points A and B, and through A, a secant is drawn intersecting the circles at the points C and D. Prove that the measure of the angle CBD is constant, i.e. it is the same for all such secants.

Problem 8 (*).
a) Show that for a prime $p > 3$ we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(p-1)^2} \equiv 0 \pmod{p}$$

b) Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1} \equiv 0 \pmod{p^2}$$