Mean Inequalities

Advanced 1

April 26, 2020

This week, we will talk about these inequalities in (two) positive variables. More concretely, for positive a, b, these inequalities are:

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

The four quantities above have the following names (in order): *harmonic mean (HM), geometric mean (GM), arithmetic mean (AM)*, and *quadratic mean (QM)*, and the three inequalities sometimes are called the *HM-GM-AM-QM* inequality. Each of this inequalities becomes an equality if and only if $a = b$ (see problem 2). Also, most often in problems, these inequalities are "hidden" somewhere, so you have to be smart in finding which variables can you use in terms of a and b. These inequalities also generalize to any set $x_1, ..., x_n$ of n positive numbers (see problem 7 for generalized AM-GM).

Problem 1.
a) Prove the AM-GM part:

$$\sqrt{ab} \leq \frac{a+b}{2}$$

b) Prove the QM-AM part

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

c) Prove the GM-HM part

$$\frac{2ab}{a+b} \leq \sqrt{ab}$$

Problem 2.
Show that each of the following follows directly from the HM-GM-AM-QM:

a) $\dfrac{x^2+y^2}{2} \geq xy$

b) $x + \frac{1}{x} \geq 2$

c) $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{x+y}$

Problem 3.
Prove that each of the HM-GM-AM-QM inequalities becomes an equality if and only if $a = b$. *Hint:* Write $b = a + x$ for some $x \neq 0$ if $a \neq b$.

Problem 4.
Suppose the product of two non-negative numbers is greater than their sum. Prove that the sum of these numbers is greater than 4.

Problem 5.
a) Show that $x^2 + y^2 + z^2 \geq xy + yz + zx$ for any x, y, z

b) Show that for all positive a, b, c, we have $(ab + bc + ca)^2 \geq 3abc(a + b + c)$.
Hint: Try to make a smart substitution and use part a).

c) Show that $x^2 + y^2 + 1 \geq xy + x + y$ for any x, y.

d) Show that for positive a, b, c, d, e we have $a^2 + b^2 + c^2 + d^2 + e^2 \geq a(b + c + d + e)$.

Problem 6.
What is the smallest value of $\frac{81 + 16x^4}{x^2}$? For which x is it achieved?

Problem 7.
a) Show the AM-GM inequality for n variables, namely that for positive $x_1, ..., x_n$ we have:

$$\sqrt[n]{x_1 x_2 ... x_n} \leq \frac{x_1 + ... + x_n}{n}$$

Proceed as follows:

1. Show (by induction) that if the inequality holds for n variables, then it also holds for $2n$ variables. Deduce it holds for all powers of 2.

2. How can you extend this result to show that the inequality holds for all n?

b) For positive $a_1, a_2, ..., a_n$ show:

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$$

c) Find all solutions to the equation $x^4 + y^4 + 2 = 4xy$.

Problem 8.
There are several blots on a square piece of paper with side A, such that each blot covers area at most 1. It so happened that each line, parallel to one of the sides of the square, intersects at most one blot. Show that the total area covered by all blots is at most A.