Lesson 2: Induction in Combinatorics

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1 Intro

Problem 1.
Prove that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Problem 2.
Fibonacci numbers, commonly denoted F_n, form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1. That is,

$$F_0 = 0, \quad F_1 = 1,$$

and

$$F_n = F_{n-1} + F_{n-2},$$

Prove that F_n is even if and only if n is divisible by 3.

2 Worksheet

Problem 3.
Show that the $2^n \times 2^n$ board with *any* square cut out can be cut into angle-trominoes like in the previous homework.

Problem 4.
$n \geq 3$ lines are drawn on a plane in such a way that no two lines are parallel and no three intersect at one point. Show that at least one of the pieces the lines cut the plane into is a triangle.

Problem 5.
Consider the following process. First, the board has the numbers $1, 1$ on it. Then between every two numbers we write in their sum, to make $1, 2, 1$. Now we do the operation for the second time, to make $1, 3, 2, 3, 1$ and so on. Show that the sum of all numbers after 100 operations will be $3^{100} + 1$.

3 Some geometry

Problem 6.
If two circles are tangent, then any secant passing through the tangency point cuts out on the circles opposed arcs of the same angular measure.

Problem 7.
Let ABC be an equilateral triangle. Show that the circle with diameter AB goes through the midpoints of BC and CA.

Problem 8.
Let ABC be a right triangle with $\angle ABC = 90°$. Suppose the circle with diameter AB intersects AC at the midpoint. Find the angles of ABC.