Lesson 3: Induction in Algebra

Konstantin Miagkov, Nikita

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Problem 1.
Show that

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

Problem 2.
Show that

$$2 + 5 + 8 + \ldots + (3n - 1) = \frac{3n^2 + n}{2}$$

Problem 3.
Show that

$$1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 4.
Show that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$$

Problem 5.
Show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(n-1) \cdot n} = \frac{n-1}{n}$$

Problem 6.
Show that

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$$

for all $n \geq 2$.

Proof. We will show this by induction. Base: $1/3 + 1/4 = 7/12 = 14/24 > 13/24$. Step: by the induction hypothesis,

$$\frac{1}{n+1} + \ldots + \frac{1}{2n} > \frac{13}{24}$$

We want to show this when n is replaced by $n + 1$:

$$\frac{1}{n+2} + \ldots + \frac{1}{2n+2} > \frac{13}{24}$$

We will in fact show that the quantity on the left increased. Indeed, let us label

$$\frac{1}{n+1} + \ldots + \frac{1}{2n} = S$$

Then

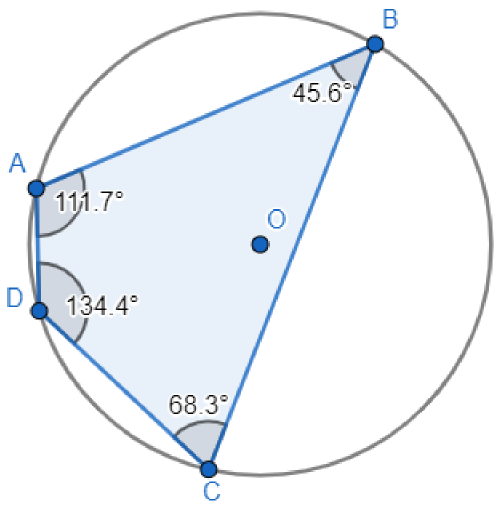
$$\begin{aligned}\frac{1}{n+2} + \ldots + \frac{1}{2n+2} &= \left(\frac{1}{n+1} + \ldots + \frac{1}{2n}\right) + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= S + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= S + \frac{2n+2+2n+1-2(2n+1)}{(2n+1)(2n+2)} \\ &= S + \frac{1}{(2n+1)(2n+2)} > S\end{aligned}$$

Therefore if $S > 13/24$, then

$$\frac{1}{n+2} + \ldots + \frac{1}{2n+2} > S > \frac{13}{24}$$

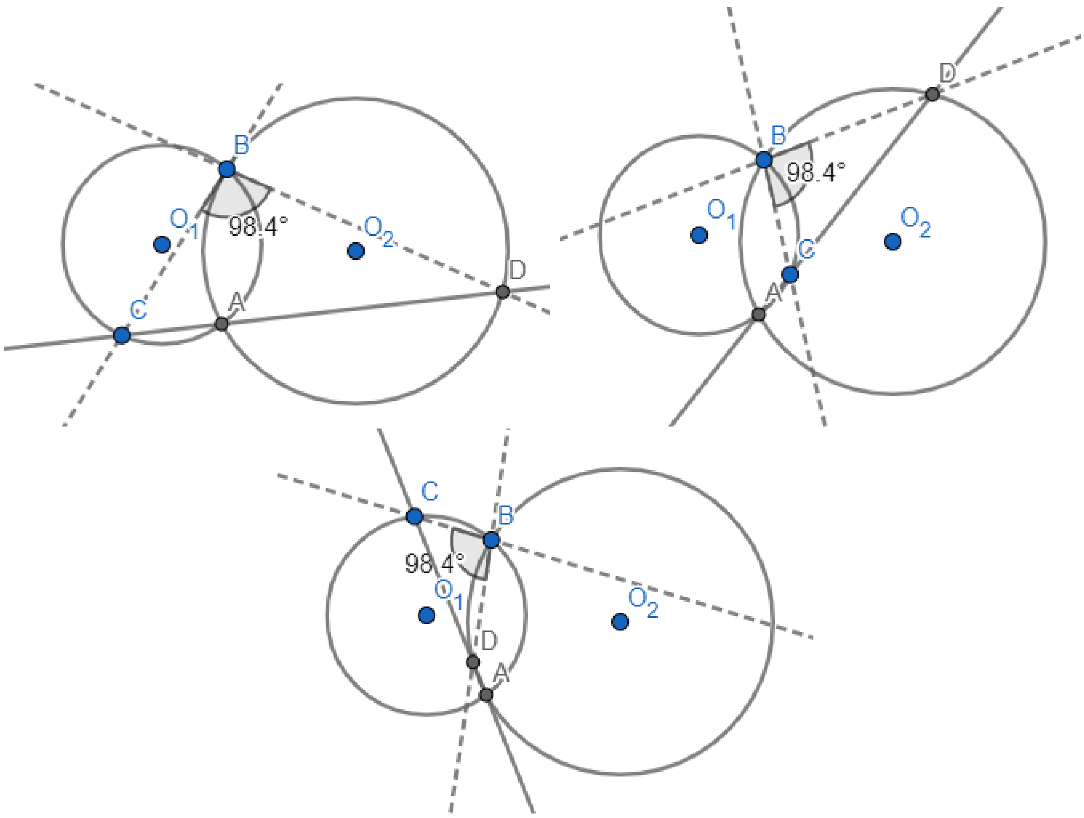
which concludes the inductive step. □

Problem 7.
Let $ABCD$ be points lying on a circle, in this order. Show that the opposite angles of the quadrilateral $ABCD$ add up to 180°.



Proof. Each angle is half of the corresponding arc, and the arcs add up to 360°. □

Problem 8.
Two circles intersect at the points A and B, and through A, a secant is drawn intersecting the circles at the points C and D. Prove that the measure of the angle CBD is constant, i.e. it is the same for all such secants.



Proof. Note that $\angle CBD = 180° - \angle BCA - \angle BDA$. Since $\angle BCA$ is half of the arc AB on the first circle and $\angle BDA$ is the same on the second circle, they do not depend on the secant and neither does $\angle CBD$. □

Problem 9.
Let PA and PB be two tangents to a circle drawn from the same point P, and let BC be a diameter. Prove that CA and OP are parallel.

Proof. Let O be the center of the circle. Then it is enough to show that $\angle ACB = \angle POB$. Notice that $\angle ACB$ is half the arc AB, and since $\triangle APO = \triangle BPO$ we also have that $\angle POB$ is half the arc AB, concluding the proof. □