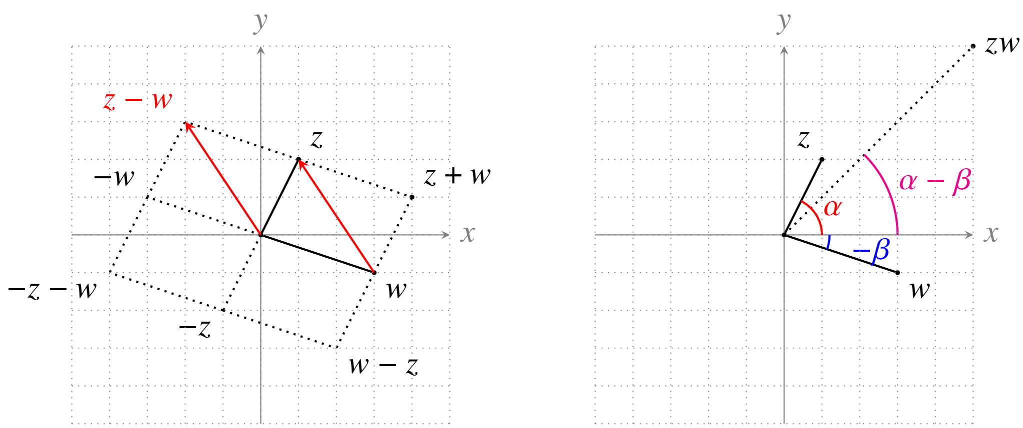
COMPLEX NUMBERS IN GEOMETRY

LAMC OLYMPIAD GROUP, WEEK 10

Last time, we defined complex numbers as $z = a + bi$, where a, b are real numbers called the *real* and *imaginary* parts of z. We can add and multiply complex numbers in the natural way, with the convention that $i^2 = -1$ (so for example, $(1 - i)(1 + i) = 1 - i^2 = 2$). We now want to use complex numbers to solve geometry problems; we'll go over this in more detail in our online sessions from Spring Quarter.

1. Recap: Operations with Complex Numbers

Recall what the addition and multiplication of complex numbers look like in the plane:



- When adding z and w, their real parts (which are the x-coordinates) add up, and their imaginary parts (the y-coordinates) add up too:

$$\text{Re}\,(z + w) = \text{Re}\,z + \text{Re}\,w, \qquad \text{Im}\,(z + w) = \text{Im}\,z + \text{Im}\,w.$$

- When multiplying z and w, their absolute values (lengths) multiply, and their arguments (angles modulo 2π with the x-axis) add up:

$$|zw| = |z| \cdot |w|, \qquad \arg(zw) = \arg z + \arg w.$$

  In the picture above, $\arg z = \alpha$, $\arg w = -\beta$, and $\arg(zw) = \alpha - \beta$.

Hence addition with a complex number is a *translation* (by $\text{Re}\,z$ in the x-direction and by $\text{Im}\,z$ in the y-direction), and multiplication by a complex number is a *scaling* (by $|z|$) plus a *rotation* (by $\arg z$).

You can think of the difference $z - w$ of two complex numbers as the vector (imagine an arrow) from w to z; see the red arrow above. Then $|z - w|$ is the distance between z and w, and $\arg(z - w)$ is the directed angle that the line from w to z makes with the x-axis. When we divide two such differences, we see that

$$\begin{aligned}\arg \frac{a - b}{c - d} &= \arg(a - b) - \arg(c - d) \\ &= \text{Directed angle}\Big(\text{Ray from } d \text{ to } c, \text{Ray from } b \text{ to } a\Big),\end{aligned}$$

for $a, b, c, d \in \mathbb{C}$ with $a \neq b$ and $c \neq d$. By *directed angle*, we mean that direction is also taken into account (so for example $\alpha \neq -\alpha$ unless $\alpha = 0$ or $\alpha = \pi = 180°$).

Also recall that we can *conjugate* complex numbers as $\overline{a+bi} = a - bi$ for $a, b \in \mathbb{R}$, and conjugation jumps over addition, subtraction, multiplication and division. We have $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2}$, so saying that a number z is real is the same as saying $\bar{z} = z$, and saying that z is (purely) imaginary is equivalent to $\bar{z} = -z$. Other properties include $|z|^2 = z \cdot \bar{z}$, and $|z+w| \le |z| + |w|$ (the triangle inequality).

2. Useful Facts about Complex Numbers in Geometry

Problem 1. (Parallelism and perpendicularity). Let a, b, c, d be complex numbers such that $a \neq b$ and $c \neq d$.

(a) Show that the line through a and b is parallel or equal to the line through c and d if and only if

$$\frac{a-b}{c-d} = \frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}.$$

(In particular, we can take $d = b$ to get a condition for when a, b, c are collinear.)

Hint: Read the previous page, and try to show that this is equivalent to saying that $\frac{a-b}{c-d}$ *is a real number.*

(b) Show that the line through a and b is perpendicular to the line through c and d if and only if

$$\frac{a-b}{c-d} = -\frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}.$$

Hint: Show that this is equivalent to saying that $\frac{a-b}{c-d}$ *is an imaginary number.*

Problem 2. (Midpoint and centroid). Let a, b, c be distinct complex numbers.

(a) Show that midpoint of the segment joining a and b is $\frac{a+b}{2}$. *Hint: Look at the x-coordinates (real parts) and y-coordinates (imaginary parts) separately.*

(b) Show that the center of mass (centroid) of the triangle with vertices a, b, c is $\frac{a+b+c}{3}$.

Problem 3. (Orthocenter). Let a, b, c be distinct complex numbers on a circle centered at 0. Show that the orthocenter of the triangle with vertices a, b, c is given by $h = a + b + c$. *Hint: Try to show that the vectors* $h - a$ *and* $b - c$ *are perpendicular using Problem 1 (b).*

Problem 4. (Equilateral triangles). Let a, b, c be distinct complex numbers. Show that the triangle with vertices a, b, c is *equilateral* if and only if

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Problem 5. (Inversion) Let z be a complex number such that $\operatorname{Re} z = 1$. Show that $1/z$ lies on the circle of center $1/2$ and radius $1/2$. *Hint: Use the Pythagorean Theorem to show that the distance between* z *and* $1/2$ *is* $1/2$.

3. Actual Geometry Problems (that can be solved via complex numbers)

Problem 6. (Ptolemy's Inequality). Show that for any quadrilateral $ABCD$, one has

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD.$$

Hint: let a, b, c, d be the complex numbers representing points A, B, C, D, and show first that

$$(a-b)(c-d) + (a-d)(b-c) = (a-c)(b-d).$$

Then take absolute values of both sides, and apply the triangle inequality (this is pure magic!).

Problem 7. In a non-equilateral triangle $\triangle ABC$, let O be the circumcenter, G the center of mass and H the orthocenter. Show that O, G and H are collinear in this order, and moreover that $GH = 2OG$.

Hint: Take the origin at O (that is, O corresponds to the complex number 0*) and use Problems 2(b), 3.*

Problem 8. (Napoleon's Theorem) Let $\triangle ABC$ be a triangle, and construct equilateral triangles $\triangle BCD$, $\triangle CAE$, $\triangle ABF$ *outside* the triangle. Show that the centers of these 3 equilateral triangles also form an equilateral triangle.

Hint: Denote the vertices of the original triangle by $a, b, c \in \mathbb{C}$, and then find explicit formulas for d, e, f and the three centers in terms of a, b, c. Then use Problem 4 from the previous section to test that the centers form an equilateral triangle.

Problem 9. Let $ABCD$ be a convex quadrilateral, and construct squares $ABEF, BCGH, CDIJ$ and $DAKL$ outside the quadrilateral. Let W, X, Y, Z be the centers of these, respectively. Show that $WY \perp XZ$ and also $WY = XZ$.

Hint: As before, denote by a, b, c, d the complex numbers representing the vertices A, B, C, D, and then compute the centers w, x, y, z in terms of a, b, c, d. Then compute $w - y$, $x - z$ and $\frac{w-y}{x-z}$.

Problem 10. In a triangle $\triangle ABC$, let H be the orthocenter and let M be the midpoint of BC. Show that the symmetric point of H with respect to M lies on the circumcircle of $\triangle ABC$.

Problem 11. (Apollonius' theorem) Let $\triangle ABC$ be a triangle and M be the midpoint of BC. Show that

$$AM^2 = \frac{AB^2 + AC^2}{2} - \frac{BC^2}{4}.$$

Problem 12. Let P be the set of n vertices of a regular n-gon with radius 1 (with $n \geq 3$; by radius we mean the radius of the circumscribed circle). Show that the average (arithmetic mean) of the squared distances between two points in P is 2 (in the average we include segments of length 0 and double count segments AB and BA). In other words, show that

$$\frac{1}{n^2} \sum_{A,B \in P} AB^2 = 2.$$