# Graph Theory and Some Topology

Aaron Anderson for Los Angeles Math Circle

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A *graph* is defined as a set $V$, whose elements are called *vertices*, together with a set $E$, whose elements, called *edges*, are pairs of vertices. Graphs are usually drawn with the vertices as dots, and the edges as line segments connecting pairs of the dots.

The following sections build in difficulty. If you are already familiar with the ideas in one section, you may skim over it and skip to the next.

## 1 Eulerian Paths, Degree

The first graph to ever be studied (according at least to the folklore) was the map of the bridges of Königsberg, Prussia (now Kaliningrad, Russia). The mathematician Leonhard Euler was asked if it was possible to take a walk around the city, crossing each bridge exactly once, and if so, could you end up back where you started?

His response was to model the city as a graph, modeling the riverbanks and islands as vertices, and the bridges as edges (technically a *multigraph*, because the same two vertices can be connected by multiple edges).

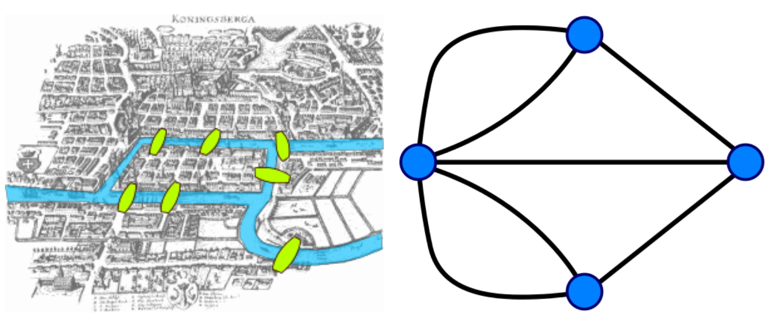

Figure 1: Königsberg in Euler's time, and the graph representing it[1]

To solve this problem, we need a few more definitions. A *path* in a graph is a sequence $v_0, e_0, v_1, e_1, v_2, \ldots, v_n$ that alternates between vertices and edges, with the edge $e_k$ connecting the vertices $v_k$ and $v_{k+1}$. Think of it as the trajectory of a person walking around the graph, using edges to get between different vertices.

An *Eulerian* path is one where each edge appears in the sequence exactly once, and an Eulerian *circuit* is an Eulerian path which begins and ends at the same vertex (there are no rules against repeating vertices in an Eulerian path).

**Problem 1** Which of the following graphs have Eulerian paths? Which have Eulerian Circuits? Try to draw the paths, and observe any patterns.

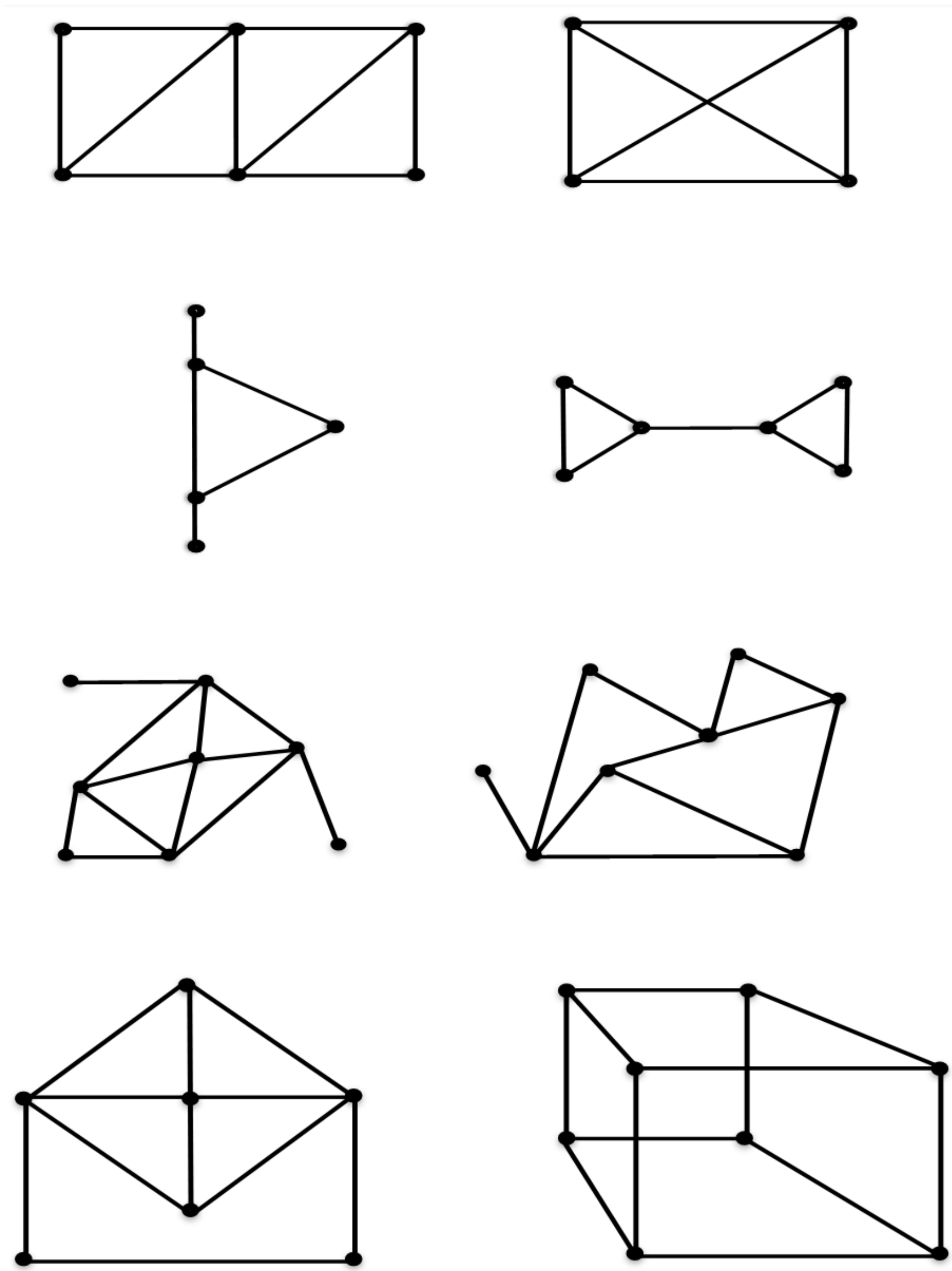

Figure 2: A few graphs[2]

To solve this in general, we need one more definition. If $v$ is a vertex in a graph $G$, then *the degree of* $v$, $\deg_G(v)$, is the number of edges with $v$ as a vertex.

**Problem 2** How many vertices of odd degree can a graph have if it has an Eulerian circuit? What if it has an Eulerian path that is not a circuit? If it does, where do the paths begin and end?

**Problem 3** Conversely, show that if a graph has the correct number of vertices of odd degree (as you determined in the previous problem), then it *must* have an Eulerian circuit/path. (Hint: Try walking around the graph randomly, except without repeating yourself.)

**Problem 4** Did Königsberg have an Eulerian path? If so, was it a circuit?

**Problem 5** **(The Handshake Lemma)** Find a formula for $\sum_{v \in V} \deg_G(v)$, that is, the sum of the degrees of all the vertices in a graph $G$ with vertex set $V$ and edge set $E$. Use it to show that the number of vertices with odd degree must be even.

## 2 Euler's Formula

In this section, we want to think of graphs not only as amorphous vertices and edges, but as having an actual shape. In particular, we care about *planar* graphs, which are those which can be drawn in the plane where the curves representing the edges do not intersect (except when they meet at vertices). Another useful definition is that of a *connected* graph: one where there is a path between every pair of vertices.

**Problem 6** These graphs are all planar, but are not drawn properly in the plane. Can you draw them so that the edges do not intersect?

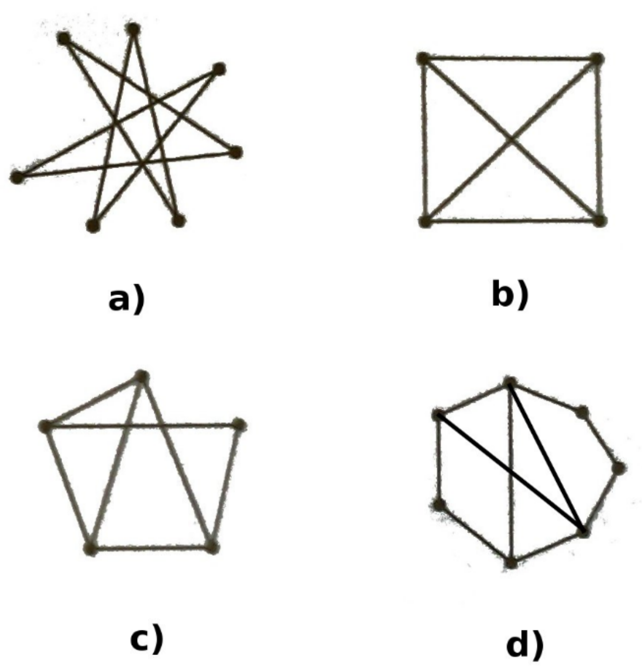


Figure 3: Some planar graphs[3]

**Problem 7** Now that you've unscrambled these graphs, notice that between the edges are 2d regions, which we call *faces*. Counting the area outside the graph as a face, let's call the number of faces of the graph $f$. While the exact set of faces, $F$, may vary depending on how you choose to draw the graph in the plane, the number $f = |F|$ does not. Where $|V|$ is the number of vertices, and $|E|$ is the number of edges, fill out the following table for the graphs from the previous problem:

| | $\|V\|$ | $\|E\|$ | $f$ | $\|V\| - \|E\| + f$ |
|---|---|---|---|---|
| a) | | | | |
| b) | | | | |
| c) | | | | |
| d) | | | | |

Can you notice and write down an equation involving $|V| - |E| + f$? This quantity is called the *Euler characteristic*, and the equation is called *Euler's formula*. We will prove later in the worksheet that it is true for all planar connected graphs.

**Problem 8** For a planar connected graph, if $|V| \geq 3$, show the following (you may assume Euler's formula):

- $2|E| \geq 3f$
- $|E| \leq 3|V| - 6$
- If the graph contains no cycles of length 3 (a cycle is a path where the first and last vertices are the same, but all other vertices are distinct; the length is the number of edges in the cycle), then $|E| \leq 2|V| - 4$.

**Problem 9** The vertices and edges of every polyhedron form a graph. Sharing the work with the rest of your group, confirm that the graphs of these five polyhedra are planar, and verify that they satisfy Euler's formula:

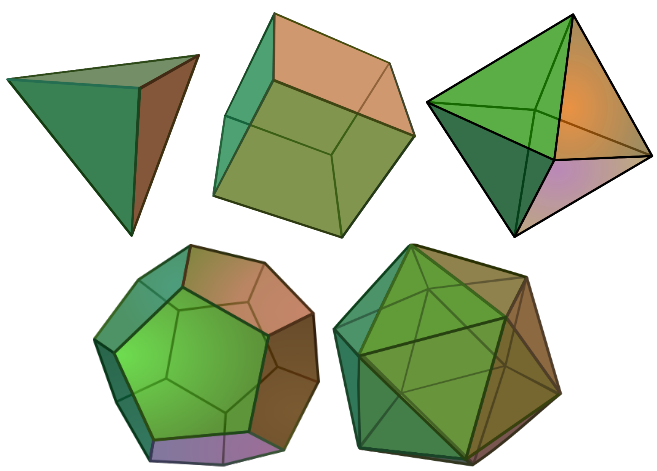

Figure 4: The Platonic Solids[4][5][6][7][8]

**Problem 10** Prove that in general, the graph of every polyhedron can be drawn in the plane with the faces (sides) of the polyhedron corresponding to the faces of the graph. Conversely, show that any planar graph can be turned into a polyhedra, using its faces as the faces of the polyhedron. Conclude that Euler's formula also holds for polyhedra.

# 3 Dual Graphs, Spanning Trees, Proving Euler's Formula

## 3.1 Dual Graphs

If $G$ is a planar graph, drawn in a specific way, then we can construct another graph, $G^*$, called the *dual graph* of $G$. The vertices of $G^*$ will be the faces of $G$, and the edges correspond directly to the edges of $G$: each edge $e$ in $G$ has a face on each side of it, let these faces be $f_1, f_2$. Then let its corresponding edge $e^*$ in $G^*$ connect $f_1$ and $f_2$.

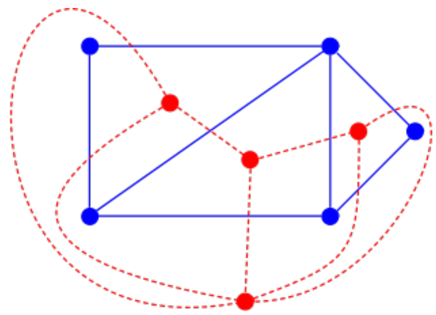

Figure 5: A planar graph and its dual[9]

**Problem 11** For each of the platonic solid graphs, find the dual of the graph, and find out what polyhedron it corresponds to.

**Problem 12** Prove that the dual of the dual of $G$, $(G^*)^*$, is the same as $G$.

## 3.2 Trees

**Problem 13** Recall that a cycle is a path where the first and last vertices are the same, but all others are distinct. Then a *tree* is a connected graph with no cycles.

**(a)** Prove that every (finite) tree must have a vertex of degree 1.

**(b)** Show by induction that every tree must satisfy $|V| = |E| + 1$. (Hint: induct on the number of vertices, and remove a piece of your tree to use your induction hypothesis.) Check that all trees satisfy Euler's formula.

**(c)** Show that every connected graph can be turned into a tree by removing all but $|V| - 1$ edges, and keeping the original vertices. This is called a *spanning tree*.

## 3.3 Proof of Euler's Formula

If $G$ is a graph with vertex set $V$ and edge set $E$, then we say another graph $H$ is a *subgraph* of $G$ when its vertex set is also $V$, but its edge set is a subset of $E$. For instance, if $T$ is a spanning tree of $G$, then $T$ is also a subgraph of $G$.

**Problem 14**

**(a)** Let $G$ be a planar graph and $G^*$ its dual. Let $H$ be a subgraph of $G$. Then let $H'$ be the subgraph of $G^*$ whose edge set consists of all edges $e^*$ where $e$ is not in the edge set of $H$. Show that $H$ has a cycle if and only if $H'$ is not connected, and that $H'$ has a cycle if and only if $H$ is not connected.

**(b)** Let $H$ be a spanning tree of $G$. Then show $H'$ is a spanning tree of $G^*$, and calculate how many edges $H$ and $H'$ have.

**(c)** Show that $G$ satisfies Euler's formula.

# 4 Sperner's Lemma

Let $T$ be a graph formed by taking the triangle with vertices $R, G, B$, and *triangulating* it, that is, breaking it up into many small triangular pieces, like so:

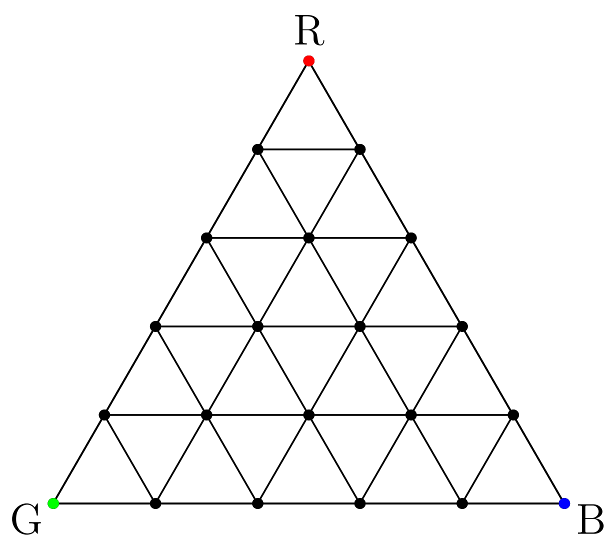


Figure 6: A sample triangulation

Now color every vertex of $T$ red, green, or blue, subject to the following rules: Color vertex $R$ red, $G$ green, $B$ blue, and every vertex on the side of the outer triangle connecting $R$ and $G$ either red or green, every vertex on the side connecting $R$ and $B$ either red or blue, and every vertex on the side connecting $G$ and $B$ either green or blue. The interior vertices can be colored with any of the 3 colors. This is called a *Sperner coloring*.

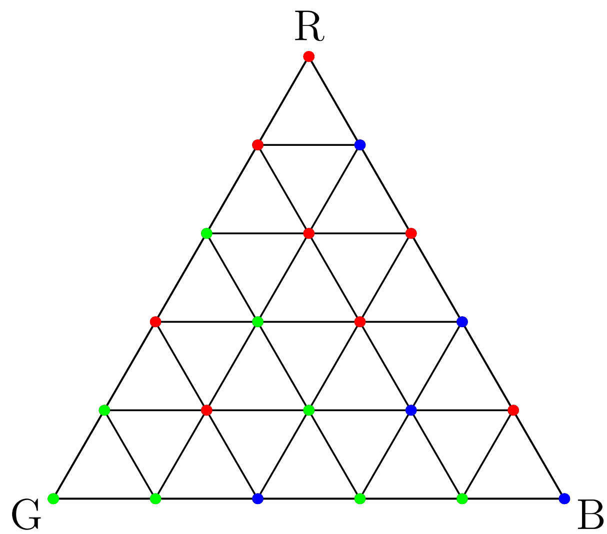


Figure 7: A Sperner coloring

The rest of this section is dedicated to proving *Sperner's Lemma*:

**Lemma 4.0.1** (Sperner)**.** *If $T$ is a triangulation colored with a Sperner coloring, at least one of small trianglular faces of $T$ will have its three vertices colored with three different colors.*

In order to prove this, we make another graph, $T'$, whose vertices are faces of $T$. All of the faces inside the triangle $RGB$ will themselves be triangles, but there will be one more face, the outside face $O$. Note that each pair of neighboring faces of $T$ meet at an edge of $T$, whose two vertices are colored. We connect these two faces, as vertices in $T'$, if and only if the two vertices of the edge where they meet are colored red and green. Faces which are not adjacent are not connected in $T'$.

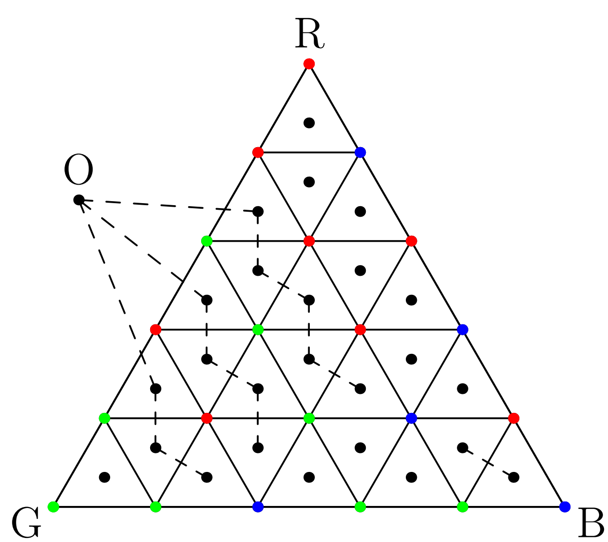


Figure 8: $T'$, with vertices in black and edges dashed

**Problem 15** In $T'$, show that the degree of $O$ is odd.

From this, show that the number of small triangles whose vertices are colored red, green, blue is odd, and thus greater than 0. This proves Sperner's Lemma. (Hint: Problem 5)

**Problem 16** What should the 1-dimensional version of Sperner's Lemma look like, or the 3-dimensional version? Prove your 1D version.

**Challenge**: Prove your 3D version of Sperner's Lemma.

# 5 Brouwer's Fixed Point Theorem

**Problem 17** Let $\Delta$ be the set of points $(x, y, z)$ in three-dimensional space where $x + y + z = 1$, and $x \geq 0, y \geq 0, z \geq 0$. Sketch a picture of $\Delta$, and describe its shape.

The Euclidean distance makes $\Delta$ a metric space, so last week's worksheet gives us a definition of when a function from $\Delta$ to $\Delta$ is continuous.

In this section, you will use Sperner's Lemma to prove Brouwer's Fixed Point Theorem about such continuous functions:

**Theorem 5.1.** *Every continuous function $f : \Delta \to \Delta$ has a fixed point.*

This is a topological fact, so in fact it is true of any shape topologically similar to $\Delta$, which includes any flat disk or rectangle, or other roughly 2D shapes like the UCLA campus, as viewed from above. In that last case, imagine that I take a campus map and put it somewhere on campus. Then this map represents a function, taking points on campus and drawing them on the map, which itself has a geographical location. Brouwer's fixed point theorem says there must be some point on the map which represents its exact location in the real world.

In order to talk in more detail about the function $f$, we generally refer to its coordinate functions, functions $f_x, f_y, f_z : \Delta \to [0, 1]$ such that for any point $P \in \Delta$, $f_x(P)$ is the $x$-coordinate of $f(P)$, $f_y(P)$ is the $y$-coordinate, and $f_z(P)$ is the $z$-coordinate. It's a fact that $f_x$, $f_y$, and $f_z$ are continuous if and only if $f$ is.

**Problem 18** Let $f : \Delta \to \Delta$ be a continuous function, and $f_x, f_y, f_z$ its coordinate functions. For $P = (x, y, z) \in \Delta$, show that either $f_x(P) \leq x$, $f_y(P) \leq y$, or $f_z(P) \leq z$. Then show that if all three inequalities are true, that $P$ is a fixed point.

The remainder of the problems are designed to show that there is a point that satisfies all 3 inequalities.

**Problem 19** Let $T$ be a triangulation of $\Delta$. Label $(1, 0, 0)$ as $R$, $(0, 1, 0)$ as $G$, and $(0, 0, 1)$ as $B$. Show that you can color $T$ with a Sperner coloring, where if a vertex $P = (x, y, z)$ is red, then $f_x(P) \leq x$, if it is green, then $f_y(P) \leq y$, and if it is blue, then $f_z(P) \leq z$. Call a Sperner coloring with this property a *Brouwer coloring.*

**Problem 20** Show that for every $n$, you can find a triangulation $T_n$ of $\Delta$ where all edges have side length at most $\frac{1}{2^n}$ times the side length of $\Delta$. Then use this to show that there are points $R_n, G_n, B_n \in \Delta$ such that the distance between any two of them is at most $\frac{1}{2^n}$, and $f_x(R_n)$ is less than or equal to the $x$-coordinate of $R_n$, $f_y(G_n)$ is less than or equal to the $y$-coordinate of $G_n$, and $f_z(B_n)$ is less than or equal to the $z$-coordinate of $B_n$.

**Problem 21** Deduce that there is a sequence $R_1, R_2, R_3, \ldots$ of points in $\Delta$ with $f_x(R_n)$, where each $R_n = (x, y, z)$ in the sequence satisfies $f_x(R_n) \leq x$, a sequence $G_1, G_2, G_3, \ldots$ where each $G_n = (x, y, z)$ satisfies $f_y(G_n) \leq y$, and a sequence $B_1, B_2, B_3, \ldots$ where each $B_n = (x, y, z)$ satisfies $f_z(B_n) \leq z$, and for each $n$, the distances between $R_n, G_n, B_n$ are at most $\frac{1}{2^n}$.

Show that if $R_1, R_2, R_3 \dots$ converges to a point $P$, then $G_1, G_2, \dots$ and $B_1, B_2, \dots$ converge to it as well. Then show $P$ is a fixed point of $f$.

(Hint: Use the definition of a sequentially continuous function.)

**Problem 22** Technically, our sequence $R_1, R_2, R_3, \dots$ may not converge, but we can find an infinite *subsequence* of it that does, by skipping some points. If the subsequence $R_{n_1}, R_{n_2}, R_{n_3}, \dots$ converges to $P$, where $n_1 < n_2 < \dots$, then show that the sequences $R_{n_1}, R_{n_2}, R_{n_3}, \dots$, $G_{n_1}, G_{n_2}, G_{n_3}, \dots$, $B_{n_1}, B_{n_2}, B_{n_3}, \dots$ still satisfy the requirements of the previous problem, so $P$ is a fixed point.

**Problem 23** Let $P_1, P_2, P_3, \dots$ be any sequence of points in $\Delta$. Show that following these steps makes $P_{n_1}, P_{n_2}, P_{n_3}, \dots$ a Cauchy sequence, which thus means that it converges to some point $P$:

1. Let $n_1 = 1$
2. Given $n_k$, break $\Delta$ into $4^k$ triangles, each with sidelength $\frac{1}{2^k}$ times the sidelength of $\Delta$.
3. Show that one of these triangles contains infinitely many points from $P_{n_k+1}, P_{n_k+2}, P_{n_k+3}, \dots$
4. Let $n_{k+1}$ be such that $P_{n_{k+1}}$ is in that triangle.
5. Repeat at Step 2

Conclude that we can indeed find an infinite subsequence of $R_1, R_2, R_3, \dots$ which converges to some point $P$, and that $P$ is indeed a fixed point, proving the theorem.

## References


[1] URL: `https://commons.wikimedia.org/wiki/File:K%C3%B6nigsberg_graph.svg`.

[2] URL: `https://circles.math.ucla.edu/circles/lib/data/Handout-702-810.pdf`.

[3] URL: `https://circles.math.ucla.edu/circles/lib/data/Handout-400-500.pdf`.

[4] URL: `https://en.wikipedia.org/wiki/File:Tetrahedron.svg`.

[5] URL: `https://en.wikipedia.org/wiki/File:Hexahedron.svg`.

[6] URL: `https://en.wikipedia.org/wiki/File:Octahedron.svg`.

[7] URL: `https://en.wikipedia.org/wiki/File:Dodecahedron.svg`.

[8] URL: `https://en.wikipedia.org/wiki/File:Icosahedron.svg`.

[9] URL: `https://commons.wikimedia.org/wiki/File:Duals_graphs.svg`.