Bertrand's postulate

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April 12, 2020

1 Heuristics of the proof:

Obviously, all primes dividing $\binom{2n}{n}$ are less than 2n. We show that for n large enough, $\binom{2n}{n}$ is big enough, the primes dividing it have small exponents, and there are no primes that divide this number and that are almost big but not big (almost big = between 2n/3 and n, and big = between n and $2n$).

2 Proof of Bertrand's postulate

Bertrand's Postulate

3 Problems

1. Let $k, n \geq 3$ be integers. Let $a = 2^k - 1$. Prove that one of the numbers $a^n + 1, a^{n+1} + 1, \cdots, a^{2n-2} + 1$ does not share any odd divisor greater than 1 with any of the other numbers.

2. Show that the sum
$$\sum_{i=1}^{n} \frac{1}{i}$$
is not an integer for $n \geq 2$.

3. Prove that
$$\frac{4^n}{2n} \geq (2n)^{\sqrt{2n}} 4^{\frac{2n}{3}}$$
for $n > a$ for some fixed a and show that Bertrand holds for $1 \leq n \leq a$. Comment: I personally choose $a = 7056$ because the inequality becomes way easier that way.

4. The integers $\{1, 2, \cdots, 2k\}$ can be arranged into k disjoint pairs so that the sums of the elements in each pair is prime.

5. Extend Bertrand's postulate to the following: For any $x \geq 1$, $x \in \mathbb{R}$, there is a prime $x < p \leq 2x$.