Rational Approximation

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April 5, 2020

1 Results

1. (Farey sequences) Let the nth **Farey sequence**

$$F_n = \left[\frac{0}{1} = \frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_N}{b_N} = \frac{1}{1}\right]$$

be the sequence of reduced rational numbers with denominator less than or equal to n. Then if $\frac{a_i}{b_i} \in F_n$ has denominator n, we have

$$\frac{a_i}{b_i} = \frac{a_{i-1} + a_{i+1}}{b_{i-1} + b_{i+1}}$$

2. (Stern-Brocot sequence) We generate **Stern-Brocot sequences** of rational numbers in $[0, 1]$ by the following rules. Set $S_1 = [\frac{0}{1}, \frac{1}{1}]$, and then if

$$S_n = \left[\frac{0}{1} = \frac{a_0}{b_0}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_N}{b_N} = \frac{1}{1}\right]$$

form S_{n+1} by inserting the **mediant** of each pair of neighbors in S_n:

$$S_{n+1} = \left[\frac{0}{1} = \frac{a_0}{b_0}, \frac{a_0 + a_1}{b_0 + b_1}, \frac{a_1}{b_1}, \frac{a_1 + a_2}{b_1 + b_2}, \dots, \frac{a_{N-1} + a_N}{b_{N-1} + b_N}, \frac{a_N}{b_N} = \frac{1}{1}\right]$$

Then every rational number appears in some S_n.

3. (Continued fraction evaluation) Let

$$[a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$$

be a continued fraction. Then if

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_n}}}$$

denotes the nth convergent, we have

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}$$

and furthermore

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$$

4. (Continued fraction approximation) Let $[a_0; a_1, a_2, \dots]$ be the continued fraction approximation of the irrational number α. Then the sequence of convergents $\frac{p_n}{q_n}$ are the best rational approximations of α in the sense that $\frac{p_{n+1}}{q_{n+1}}$ is the fraction $\frac{p}{q}$ of least denominator such that

$$|q\alpha - p| < |q_n\alpha - p_n|$$

5. (Dirichlet) If α is irrational, then for all N there exists $n \leq N$ such that

$$|n\alpha - m| < \frac{1}{N}$$

As a corollary, there exist infinitely many $\frac{m}{n}$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}$$

6. (Hurwitz) If α is irrational, then there exist infinitely many fractions $\frac{m}{n}$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{\sqrt{5}n^2}$$

Furthermore, the constant $\sqrt{5}$ cannot be increased, since if $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{1-\sqrt{5}}{2}$, then the best approximations of ϕ are given by $\frac{F_{n+1}}{F_n}$, where

$$F_n = \frac{\phi^n - \overline{\phi}^n}{\sqrt{5}}$$

and

$$|F_n\phi - F_{n+1}| = \phi|F_n + \overline{\phi}F_{n+1}| = \phi|\overline{\phi}^{n+1}| = \frac{1}{\phi^n} = \frac{1}{\sqrt{5}F_n + \overline{\phi}^n}$$

7. (Thue-Siegel-Roth) For all irrational algebraic numbers α and $\epsilon > 0$, there exists a constant C such that the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{C}{n^{2+\epsilon}}$$

has no solutions $\frac{m}{n}$.

2 Problems

1. (Simultaneous Dirichlet) Prove that if $\alpha_1, \alpha_2, \ldots \alpha_d$ are d irrational numbers, then for all N, there exists $n \leq N^d$, $\{m_i \in \mathbb{Z}\}$ such that

$$|n\alpha_i - m_i| < \frac{1}{N}$$

for all i. Deduce that there exist infinitely many n such that

$$\left|\alpha_i - \frac{m_i}{n}\right| < \frac{1}{n^{1+\frac{1}{d}}}$$

holds for all i.

2. Find the continued fraction approxmation of $\sqrt{2}$.

3. Find the continued fraction approximation of e.

4. Explain why plants tend to grow their nth leaves at an angle of $2\pi(2-\phi)$ from the $n-1$th leaf (see http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html).