PERMUTATIONS

JUNIOR CIRCLE 04/24/2011

Several people (with labels $1, 2, 3 \ldots$ attached to the back of their T-shirts) are standing in places $1, 2, 3 \ldots$ at the beginning. Suppose that they switch places. For example, let's say that

- person from place 1 moves to place 3;
- person from place 2 remains in the same place;
- person from place 3 moves to place 1.

We can write this as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix}.$$

The first row represents the starting positions. The second row represents the end positions. To make the notation shorter, we will not be drawing arrows, so that the above looks like

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

This is called a *permutation* because we are permuting (or switching around) the original positions.

(1) Suppose that we have the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

(a) What position did not shift?

(b) Where did the first position shift to?

(c) What position moved into position 3?

(2) In a row of 5 people, the middle person stayed in his place. The two people at the ends of the row switched places. Their neighbors also switched places. Indicate where each of the positions moved to:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix}.$$

(3) In a row of 5 people, all of the odd positions didn't change. The two even positions switched. Indicate where each of the positions moved to:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix}.$$

(4) In a row of 6 people, the first 5 people shifted one place to the right, and the last person moved to the first place. Write this down as a permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & & \end{pmatrix}.$$

(5) In a row of three people,
- first, people in places 1 and 2 exchanged places;
- second, people in places 2 and 3 exchanged places;

Model this situation by using numbered squares of papers. Then, write down the resulting permutation:

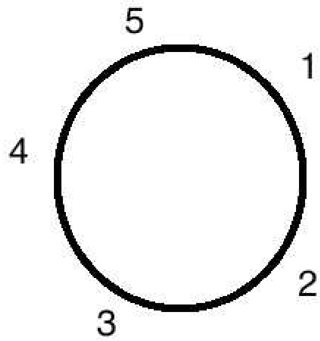
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$

(6) In a row of four people,
- first, the people in places 2 and 3 exhanged places;
- second, people in places 4 and 2 exchanged places;
- finally, people in places 1 and 3 exhanged places;

Model this situation by using numbered squares of papers. Then, write down the resulting permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}.$$

(7) There are 5 people sitting around the table.



(a) They get up and move one place to the left. Write down the permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix}.$$

(b) They get up and move one place to the right. Write down the permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix}.$$

(c) Can you interpret this result? (Recall that the first permutation means shifting by 1 to the left and the second means shifting by 1 to the right).

(8) Compare the following two permutations. Which of them would you call more "mixed-up"? Why?

- The first permutation is
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix};$$
- The second permutation is
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix};$$

(9) Play the following game several times:
- select several cards (e.g., 4);
- place them on the table in the usual order $(1, 2, 3, 4)$;
- exchange two cards (e.g., switch 2 and 4);
- exchanged two other cards;
- repeat the previous step several times (as many as you want to);
- record the permutation you get below:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}.$$

(10) Play the following game with your partner:
- Take 4 cards with numbers $1, 2, 3, 4$ written on them;
- mix up the order of the cards in some way (any way you like);
- ask your partner to get this permutation from to original order by perfoming several switches.
- Do you think you can always do this?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}.$$

(11) How can we get the follwing permutation by performing several switches?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 1 & 3 \end{pmatrix};$$

- Model this with the following:
  - First, switch positions $\square$ and $\square$;
  - Second, switch positions $\square$ and $\square$.
- Can you find another way?
  - First, switch positions $\square$ and $\square$;
  - Second, switch positions $\square$ and $\square$.

(12) In problem 11, we performed one permutation followed by another permutation. Let's do something like this again.

- Suppose you first perform the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix}.$$

- Then, you perform the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}.$$

- Model this with cards to figure out where the positions move after the two switches:

$$1 \to \square \to \square$$

$$2 \to \square \to \square$$

$$3 \to \square \to \square$$

- If we forget about the intermediate step, this can be written as:

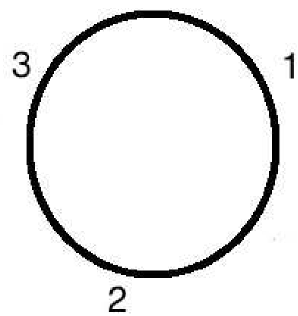
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$

- To show the result of these two permutations performed one after the other we write:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$

(13) Find the result of performing the following two permutations in a row:

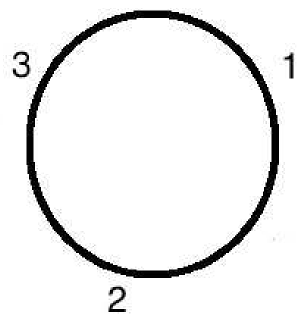
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$



- If 3 people are sitting around the tables, what does the first permutation represent?
- What does the second permutation represent?
- What is the result of doing these 2 operations one after the other?
- Does this agree with your answer above?

(14) Find the result of performing the two permutations in a row:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$



- If 3 people are sitting around the tables, what does the first permutation represent?
- What does the second permutation represent?
- What is the result of doing these 2 operations one after the other?
- Does this agree with your answer above?

(15) Write down a permutation of 3 numbers:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$

Then write it down as a sequence of two switches:

- First, switch positions $\square$ and $\square$.
- Second, switch positions $\square$ and $\square$.