Fixed Points

1 Basic Examples

Definition: Let X be a set and $f : X \to X$ a function from X to itself. A *fixed point* of f is an element $x \in X$ with $f(x) = x$.

Fixed points are very useful in various areas of mathematics. One of the most fundamental questions we may ask about a function $f : X \to X$ is whether or not it has any fixed points. We may also wonder - does it have more than one?

1.1 1D Problems

Problem 1:

a) Find all fixed points of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

b) Find all fixed points of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x + 1$.

c) Under what conditions does a quadratic function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2 + bx + c$ have a unique fixed point? When does it have more than one, and when does it have exactly one? Try to give a graphical description as well as an algebraic one.

Problem 2: Find all fixed points of the following functions.

a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + x^2 - 1$

b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 3x^2 + 4x - 1$

c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - x^2 + 2x - 1$

Problem 3: Explain graphically why $f : [-1, 1] \to [-1, 1]$ given by $f(x) = 1 - |x|$ has a unique fixed point. What is it?

1.2 2D Problems

Problem 4: Find all fixed points of the following functions.

a) $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ (sending $a + bi$ to $(a + bi)^2$)

b) $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \overline{z}$ (complex conjugation, sending $a + bi$ to $a - bi$)

c) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (3x + 2y, y)$

d) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y, 2x)$

e) Under what conditions does $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (ax + by, cx + dy)$ have a unique fixed point?

Functions in the form of problem 4e are called *linear transformations*, and nonzero fixed points of a linear transformation are *eigenvectors* of the linear transformation corresponding to *eigenvalue* (scaling factor) 1. Eigenvectors (possibly with different scaling factors other than 1) are very important in the study of linear transformations. They let us talk about high dimensional transformations in much simpler ways (namely, scaling certain directions). [1]

[1] If you walk randomly on a graph (a set of vertices with edges between them), you can use fixed points (in particular, eigenvectors of eigenvalue 1) to approximate the probability that you are at a given vertex after a million steps.

2 Roots and the IVT

As you've seen through the various examples in section 1.1, there is an easy back and forth between finding the fixed points of some function $f : \mathbb{R} \to \mathbb{R}$ (solutions to $f(x) = x$) and finding roots of some other function $g : \mathbb{R} \to \mathbb{R}$ (solutions to $g(x) = 0$). We will often be able to exploit this back and forth. If we are interested in finding roots, we may instead rephrase the problem as finding fixed points. Alternatively, other times, it might be easier to find fixed points by finding roots of some other function.

Since we are often concerned with whether a function has fixed points, it is convenient to know that there is a theorem that can help guarantee that a function has roots. We can use this in our search for fixed points.

Definition: Let X, Y be metric spaces. A function $f : X \to Y$ is said to be *sequentially continuous* if whenever we have a sequence $(x_n)_{n=1}^{\infty}$ converging to some $x \in X$, then the corresponding sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(x)$ in Y.

Remark: This isn't the standard definition of continuity, but it is equivalent in the case of metric spaces. So, we will refer to sequentially continuous functions simply as continuous functions.

Don't worry too much about this definition at the moment. All you need to know is that "standard" functions $f : \mathbb{R} \to \mathbb{R}$, namely, polynomials, exponentials, $\sin(x)$, $\cos(x)$, their products, sums, and compositions, are all continuous.

Theorem: (Intermediate Value Theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose $f(a)$ and $f(b)$ have opposite signs (i.e. $f(a)f(b) < 0$) for $a < b \in \mathbb{R}$. Then there is a real number $c \in \mathbb{R}$ with $a < c < b$ and $f(c) = 0$. (Recall such a $c \in \mathbb{R}$ is called a *root* or *zero* of f).

We'll prove this in challenge section 4.

Problem 5: Interpret continuity and the IVT graphically.

Problem 6: Prove $f(x) = \sin(x) + 3x^3 + 1$ has a root.

Problem 7: Prove that the graphs of the function $f(x) = x^7 + \sin(x)$ and the function $g(x) = -x + 1$ must intersect.

Problem 8:

a) Prove $f(x) = x^5 + x^3 + 2x - 1$ has a fixed point.

b) In fact, show that this fixed point is unique.

Sadly, this method of showing the existence of fixed points won't generalize to $\mathbb{R}^2$ or higher dimensions, nor will it let us work in other more general metric space.

3 Contractions

3.1 Uniqueness

Let (X, d) be a metric space. Recall:

Definition: A function $f : X \to X$ is said to be a *contraction* by a factor $0 \leq c < 1$ if for any pair of points $x, y \in X$, we have $d(f(x), f(y)) \leq c \cdot d(x, y)$.

Problem 9: Prove contractions are continuous (see the definition in the previous section).

Problem 10: Determine which of the following functions are contractions. For each contraction, determine the (smallest) contraction factor c.

a) (X, d) any metric space, $x_0 \in X$ a fixed element, $f : X \to X$ the constant function given by $f(x) = x_0$ for all $x \in X$

b) $\mathbb{R}^2$ equipped with the Euclidean metric, $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (\frac{x}{2}, \frac{y}{3})$

c) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (\frac{x}{2} + \frac{1}{6}, \frac{y}{3} + \frac{1}{3})$

d) $f : [0, 1] \to [0, 1]$ given by $f(x) = x^2$

e) $f : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ given by $f(x) = x^2$

Problem 11:

a) Let $f : X \to X$ be a contraction by a factor $0 \leq c < 1$. Let $x, y \in X$ be two fixed points of f, i.e. $f(x) = x$ and $f(y) = y$. Show $x = y$.

b) Find all fixed points for each function in the previous problem.

c) Using parts a and b, give an alternative argument that the function in $15d$ is not a contraction.

Problem 12:

a) Let $f : [a, b] \to [a, b]$ be a continuous function. Show f must have a fixed point.

b) Verify algebraically that the function $f : [0, 1] \to [0, 1]$ given by $f(x) = \frac{x^3}{3} - \frac{x}{2} + 1$ is a contraction. Conclude it has a *unique* fixed point.

c) Verify that $f : (0, 1) \to (0, 1)$ given by $f(x) = x/2$ is a contraction; yet, it has no fixed points.

3.2 Location

Problem 13: Pick a number, any number, any number at all. Take the cosine of this (real) number, then divide it by 2. Then, take the cosine of that answer, and divide by 2 again. Do this until your answer stabilizes. What do you get? Compare with your peers.

We've stumbled on a more general phenomena. It turns out that if we iterate a contraction, we always approach the fixed point (if there is one).

Problem 14: Let $f : X \to X$ be a contraction by a factor $c \in [0, 1)$. Let $x^* \in X$ be a fixed point, i.e. with $f(x^*) = x^*$.

a) Show $f \circ f : X \to X$, which sends x to $f(f(x))$, is a contraction by a factor of $c^2 < c < 1$. That is, applying f twice contracts even more!

b) Starting with any $x_1 \in X$, we can keep applying the function f. In this way, we get $x_2 = f(x_1)$, $x_3 = f(x_2)$, and more generally, $x_{n+1} = f(x_n)$ for each $n \geq 1$. Show that the sequence $x_1, x_2, x_3, ...$ converges to the fixed point x^*.

Remark: This gives another argument that fixed points for contractions are unique: we showed that this sequence must converge to an arbitrary fixed point of f, so that it must converge to every fixed point of f. However, a sequence can only converge to one point!

Problem 15: Numerically approximate the unique fixed point of the function from 12b.

Problem 16: Visually illustrate the fixed point iteration (i.e. the technique from problem 14b) for the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \frac{z-1}{2}$.

Problem 17: (Optional: Calculus) One fixed-point iteration technique that you might already be familiar with is *Newton's Method* for finding the root of a differentiable function. For the sake of simplicity, let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with exactly one root. Pick any $x_0 \in \mathbb{R}$. Then, for successive iterations, set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This ought to get us closer to the root of f. What is the function $g : \mathbb{R} \to \mathbb{R}$ that Newton's method is trying to find a fixed point for? Verify that any fixed point of g will give us a root of f, and conversely, any root of f is a fixed point of g. Thus, if g is a contraction, Newton's method is guaranteed to converge to the unique root of f.

3.3 Existence

As we learned, when we run fixed-point iteration for a contraction map with a fixed point, we get closer and closer to the fixed point (more formally, the sequence converges to the fixed point). But what if the sequence has nowhere to converge to? The iterates will still cluster together, near each other, but might not have anywhere to converge to.

As a simple example, $f : (0, 1) \to (0, 1)$ given by $f(x) = x/2$ (see 12c) is a contraction. If we run fixed point iteration starting with any $x_1 \in (0, 1)$, we will get $x_n = x_1/2^{n-1}$. In $\mathbb{R}$, this sequence would converge to 0, but in $(0, 1)$, there is no 0. At the very least, this sequence remains *Cauchy*.

Definition: A sequence $(x_n)_{n=1}^{\infty}$ in a metric space X is *Cauchy* if, given any radius $r > 0$, we may center a ball of radius r at some point in X so that it contains almost all of the sequence (all but finitely many terms in the sequence).

Recall every convergent sequence is Cauchy, but not every Cauchy sequence converges.

Definition: A metric space in which every Cauchy sequence converges is called a *complete* metric space.

Theorem: $\mathbb{R}^n$ with the standard Euclidean metric is a complete metric space.

By contrast, $\mathbb{Q}, (0, 1), \mathbb{R} \setminus \{0\}$ are not complete (each equipped with the standard Euclidean metric).

Problem 18: (Challenge) Let $A \subset \mathbb{R}$. Equip A with the standard Euclidean metric. Then A is complete if and only if $A^c \subset \mathbb{R}$ is the union of open intervals.

Finally, we are at the main theorem that will let us guarantee fixed points.

Theorem: Let X be a complete metric space and let $f : X \to X$ be a contraction by a factor $c \in [0, 1)$. Then f must have a fixed point. (By previous observations, it must be unique).

We will prove this theorem by trying to run fixed point iteration, hoping it will converge to the fixed point. But, since we are working in a complete metric space, it will be enough to check that we get a Cauchy sequence!

Problem 19: Let $x_1 \in X$ be arbitrary. Let $x_{n+1} = f(x_n)$ for all $n \geq 1$. Let $M = d(x_2, x_1)$.

a) Prove $d(x_3, x_2) \leq cM$.

b) Prove $d(x_4, x_3) \leq c^2 M$. More generally, show $d(x_{n+2}, x_{n+1}) \leq c^n M$. So, each term in the sequence is very close to the next one (because $0 \leq c < 1$, so this is exponetnially decaying distance). However, there is more work to do, as we want arbitrary terms in the sequence to be close, not just consecutive ones.

c) (Optional) Show by induction or otherwise for any $n \geq 1$ that $1 + c + ... + c^{n-1} = \frac{1-c^n}{1-c}$ for all $n \geq 1$. Notice $\frac{1-c^n}{1-c} \leq \frac{1}{1-c}$. Similarly, by factoring out a c^{m-1}, show for $1 \leq m < n$, $c^{m-1} + ... + c^{n-2} \leq \frac{c^{m-1}}{1-c}$.

d) Show for any $n > m \geq 1$

$$d(x_n, x_m) \leq c^{m-1} \frac{M}{1-c}$$

So, even the arbitrary terms are close (since the c^{m-1} term still causes exponential decay). *Hint*: Apply triangle inequality repeatedly to get $d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-2}, x_{n-3}) + ... + d(x_{m+1}, x_m)$. Then use part *b*. Then use part *c*.

e) Let $r > 0$. We need to find a center for a ball of radius r so that it contains almost the entire sequence. Where should we center this ball? *Hint:* Pick m very large in part d so that centering the ball at x_m ensures all following terms are inside the ball.

f) Part *e* lets us conclude the sequence is Cauchy. Since X is complete, it converges to some $x^* \in X$. We have $(x_n)_{n=1}^{\infty}$ converges to x^*. On the other hand, use problem 9 and the definition of continuity to conclude this sequence also converges to $f(x^*)$. Thus, $x^* = f(x^*)$, and f has a fixed point!

4 Challenge: Proving IVT

Let X be a metric space. We say $U \subset X$ is open if for every $x \in U$, there is a ball centered at x contained entirely in U. That is, $B(x, r) \subset U$ for some $r > 0$. In particular, $U = \emptyset$ is open, as is $U = X$.

Problem 20: Show the only open subsets of $\mathbb{R}$ are the empty set and unions of open intervals.

Problem 21: Let $f : X \to Y$ be continuous as in the definition in section 2. Show that for any open subset $U \subset Y$, we have $f^{-1}U := \{x \in X : f(x) \in U\}$ is open in X.

Definition: A metric space is said to be *connected* if it cannot be written as the union of two nonempty disjoint open sets.

Problem 22: Show the only connected subsets of $\mathbb{R}$ (with the standard metric) are intervals of the form (a, b), $[a, b]$, $[a, b)$, or $(a, b]$, where a, b may possibly be taken to be $\pm\infty$. Here, when considering whether $S \subset \mathbb{R}$ is connected, give S the standard Euclidean metric to make it a metric space and apply the previous definition.

Problem 23: Let $f : X \to Y$ be continuous. Show that if X is connected, then $f(X) \subset Y$ is connected (as in the previous problem, give $f(X)$ the metric of Y so that it is a metric space).

Problem 24: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show $f(\mathbb{R})$ is connected, and conclude the intermediate value theorem.