REVIEW OF WINTER 2020

LOS ANGELES MATH CIRCLE
ADVANCED 2
APRIL 5, 2020

1. Metric Spaces

Problem 1.1. Let $X = \mathbb{R}^2$, and let d_T and d_E be the Taxicab and Euclidean metric respectively. Show that there is some constant c such that for any two points $x, y \in \mathbb{R}^2$, we have $d_T(x, y) \leq c \cdot d_E(x, y)$. On the other hand, show that we always get $d_E(x, y) \leq d_T(x, y)$.

Problem 1.2. Let $X = \mathbb{N}$ with $d_X(a, b) = \left|\frac{a}{b} - \frac{b}{a}\right|$. Is d_X a metric on X? Similarly, let Y be the set of all subsets of a finite set Z, and let $d_Y(A, B) = |A \setminus B| + |B \setminus A|$. Is d_Y a metric on Y?

Recall: Let (X, d) be a metric space. Recall the open ball of radius $r > 0$ centered at $x_0 \in X$ is the set

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}.$$

(i.e. the set of all points at distance at most r from x_0).

Problem 1.3. Let $X = \mathbb{R}^2$ be equipped with the taxicab metric d_T. Draw $B(0, 1)$.

Problem 1.4. Let X be the set of all strings of length 4 consisting of the letters a through z. Equip X with the Hamming distance. How many strings are in $B(dogs, 3)$, $B(dogs, 2)$, and $B(dogs, 1)$?

Recall: An infinite sequence $x_1, x_2, x_3, ...$ (denoted $(x_n)_{n=1}^{\infty}$) of elements of a metric space X converges to a point $x \in X$ if every open ball centered at x contains x_n for all but finitely many n. "No matter how small, every ball centered at x contains *almost all* of the sequence."

Problem 1.5. Let x and y be points in a metric space (X, d) and suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in X which converges to both x and y. Show that $x = y$.

Problem 1.6. Describe all sequences that converge to *dogs* in the metric space from problem 1.4. Describe all sequences that converge simultaneously to *dogs* and *cats*.

Recall: A sequence $(x_n)_{n=1}^{\infty}$ in a metric space X is *Cauchy* if for any given radius, we can find a place to center a ball of that radius which contains almost the entire sequence. (Compare this to when a sequence converges to $x \in X$, where we only consider balls centered at x.)

Problem 1.7. Suppose a sequence $(x_n)_{n=1}^{\infty}$ converges to $x \in X$. Show that the sequence $(x_n)_{n=1}^{\infty}$ must be Cauchy. Conversely, give an example of a metric space and a Cauchy sequence in it which does not converge to anything.

Problem 1.8. Let (X, d) be a metric space. Let $f : X \to X$ be a function such that $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$ for all $x, y \in X$. (Such a function is called a *contraction* by a factor of $\frac{1}{2}$). Suppose further that there is a point $x \in X$ with $f(x) = x$. Show that there is no other point with this property.

2. Probability

Problem 2.1. You roll a fair four-sided die. If the result is 1 or 2, you roll again, but otherwise you stop. Calculate the probability that the sum of all your rolls is at least 4.

Problem 2.2. Let X and Y be independent random variables. Prove that $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$.

Problem 2.3. Let X be a probability space with probability measure $\mathbb{P}$. Let $A, B \subset X$ be events with nonzero probability. Prove $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$

3. Primes in Extensions of the Integers

Recall: In the extension $\mathbb{Z}[\sqrt{d}]$, an element x is said to be **irreducible** if $x = yz$ implies that either y or z is a unit. An element x is called **prime** if x divides yz implies that either x divides y or x divides z.

Problem 3.1. Is 7 an irreducible element of $\mathbb{Z}[\sqrt{-2}]$?

Problem 3.2. Is 73 an irreducible element of $\mathbb{Z}[\sqrt{6}]$?

Problem 3.3. Prove that 2 is irreducible but not prime as an element of $\mathbb{Z}[\sqrt{3}]$.

4. Generating Functions

Problem 4.1. Find a closed form for the generating function for the sequence $1, 2, 3, 4, 5, 6, \ldots,$ i.e. the function $f(x) = \sum_{n=1}^{\infty} nx^n$

Problem 4.2. The triangle numbers are the sequence $1, 3, 6, 10, \ldots$ given by the formula $t_n = \sum_{k=1}^{n} k$. The generating function for the triangle numbers is $T(x) = \sum_{n=1}^{\infty} t_n x^n$. Find a closed form for $T(x)$ (it should be a rational function).

Problem 4.3. The Catalan numbers are the sequence of numbers $c_0, c_1, c_2, \cdots = 1, 1, 2, 5, 14, 42, \ldots$ where c_n is the number of strings of n left and n right parentheses that are correctly matched, with each (being matched to a unique) that comes later in the string. That is, strings like (()) or ()() are allowed, but)()(is not allowed because it starts with a) not preceded by a (, and ())(is also not allowed because the two)s must each be matched to an earlier (, but there is only one of those.

- Show that $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$ for $n \geq 0$.
- Use that recurrence relation to find a closed form expression for the generating function $C(x) = \sum_{n=0} c_n x^n$. You want to express it as $\frac{p(x) \pm \sqrt{q(x)}}{r(x)}$, where p, q, r are polynomials. You can resolve the $\pm$ by calculating $C(0)$ from $C(x) = \sum_{n=0} c_n x^n$, and seeing whether a $+$ or a $-$ makes sense with that value. To do this, you can either use a graph (calculator/Desmos allowed) or calculus.

5. Challenge

Problem 5.1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X, d) and suppose that $(x_n)_{n=1}^{\infty}$ converges to $x \in X$. Show that for any $y \in X$, the sequence of real numbers $(d(x_n, y))_{n=1}^{\infty}$ converges to $d(x, y)$. (As usual, we consider $\mathbb{R}$ as a metric space with metric $d_{\mathbb{R}}(a, b) = |a - b|$ for all $a, b \in \mathbb{R}$.)

Problem 5.2. Let R be a commutative ring. Let $I \subset R$ be a proper subset.

a) Set up a relation on R given by $x \sim y$ if and only if $x - y \in I$. Find necessary and sufficient conditions on the subset I so that this is an equivalence relation.
b) Let I satisfy the conditions above. Let R/I be the set of all equivalence classes under the equivalence relation. Find necessary and sufficient conditions on I so that multiplication and addition are still well-defined. (This will give R/I the structure of a commutative ring). Such a subset I is called an *ideal* of R.
c) Find necessary and sufficient conditions on an ideal I so that no two nonzero elements of R/I multiply to 0. Such an ideal is called a *prime* ideal.
d) Can you relate the notion of a prime ideal to a prime element from section 2?