Point Mass Geometry II

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Warm-up problems

Use point mass geometry to solve the following problems.

1. The base $ABCD$ of a pyramid $FABCD$ is a parallelogram. The plane α intersects AF, BF, CF and DF at points A_1, B_1, C_1, D_1 respectively. Given that
$$\frac{|AA_1|}{|A_1F|} = 2, \quad \frac{|BB_1|}{|B_1F|} = 5, \quad \frac{|CC_1|}{|C_1F|} = 10,$$
find the ratio $r = \frac{|DD_1|}{|D_1F|}$.
Let $O = AC \cap BD$. Let K be the point of intersection of FO and the plane α. Use the following plan to solve the problem.

   (a) Notice that K is the point of intersection of FO and A_1C_1.

   - Place point masses at A and F in such a way that A_1 is their center of mass.
   - Place point masses at F and C in such a way that C_1 is their center of mass.
   - Point K is the center of mass of the system of these 4 masses (two of which are placed at F). Use this to find the ratio $|FK| : |KO|$.

   (b) Notice that K is also the point of intersection of FO and B_1D_1. Let r be the ratio $r = \frac{|DD_1|}{|D_1F|}$.

   - Place point masses at points B and F in such a way that $B_1$is their center of mass.
   - Place point masses at F and D in such a way that D_1 is their center of mass.
   - Point K is the center of mass of the system of these 4 masses (two of whcih are placed at F). Use the ratio $|FK| : |KO|$ that you found in part (a) of the problem to determine r.

Solution:

a) Put masses 1A, 2F, 1C, 10F so that A_1,C_1 are the respective centers of mass. Then K is the center of mass of this system, which is computed using $2O$ and $12F$ so we have that $|FK| : |KO| = 1 : 6$.

b) Put masses 1B, 5F, 1D, rF, giving us $2O, (5+r)F$. By the previous part, this must result in a ratio $|FK| : |KO| = 1 : 6$, so therefore $2 : 5 + r = 2 : 12$, giving us $r = 7$.

2. Let $L \in AC$ and $M \in BC$ be the points on the sides of $\triangle ABC$ such that

$$|CL| = \alpha \cdot |CA|, \quad |CM| = \beta \cdot |CB|, \qquad \text{where } 0 < \alpha, \beta < 1).$$

Let $P = AM \cap BL$. Find the ratio $\frac{|AP|}{|AM|}$.

Solution: Assign masses $1C$, $\frac{\alpha}{1-\alpha}A$, $\frac{\beta}{1-\beta}B$ which gives the right points L and M. Then we have that $\frac{|AP|}{|AM|} = \frac{\frac{1}{1-\beta}}{\frac{\alpha}{1-\alpha}+\frac{1}{1-\beta}} = \frac{1-\alpha}{1-\alpha\beta}$.

Center of mass via vectors

- Let $Z = Z(m_1A,\ m_2B)$ be the center of mass of point masses m_1 and m_2 placed at points A and B. Then the law of levers $m_1d_1 = m_2d_2$ can be written as
$$m_1|\overrightarrow{ZA_1}| \;=\; m_2|\overrightarrow{ZA_2}|.$$
- Since the vectors $\overrightarrow{ZA_1}$ and $\overrightarrow{ZA_2}$ have opposite directions, it follows that
$$m_1\overrightarrow{ZA_1} + m_2\overrightarrow{ZA_2} = 0.$$

  This condition can be taken as the definition of the center of mass of the system of two points. In the case when there are more then two masses, we get the following

Definition.

The center of mass of the system of n point masses $m_1A_1, m_2A_2, \dots m_nA_n$ is the point Z such that

$$m_1\overrightarrow{ZA_1} + m_2\overrightarrow{ZA_2} + \ldots + m_n\overrightarrow{ZA_n} = 0.$$

1. Show that if $Z = Z(m_1A_1, m_2A_2)$, then for any point O we have

$$\overrightarrow{OZ} = \frac{m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2}}{m_1 + m_2}.$$

Solution: We have that $\overrightarrow{OA_1} = \overrightarrow{OZ} + \overrightarrow{ZA_1}$ and similarly $\overrightarrow{OA_2} = \overrightarrow{OZ} + \overrightarrow{ZA_2}$. Then substituting this in to $\frac{m_1\overrightarrow{OA_1}+m_2\overrightarrow{OA_2}}{m_1+m_2}$, we get $\frac{(m_1+m_2)\overrightarrow{OZ}+m_1\overrightarrow{ZA_1}+m_2\overrightarrow{ZA_2}}{m_1+m_2} = \frac{(m_1+m_2)\overrightarrow{OZ}+0}{m_1+m_2} = \overrightarrow{OZ}$ since Z is the center of mass of m_1A_1, m_2A_2.

2. Suppose that for any point O on the plane and a point Z the equality above holds. Is it true that Z is the center of mass of m_1A_1 and m_2A_2?

Solution: Choose $O = Z$, then we get that $\overrightarrow{0} = \overrightarrow{ZZ} = \frac{m_1\overrightarrow{ZA_1}+m_2\overrightarrow{ZA_2}}{m_1+m_2}$, and so multiplying by $m_1 + m_2$ we see that Z is the center of mass.

3. Let G be the point of intersection of medians of $\triangle ABC$. Find $\overrightarrow{AG}$ in terms of $\overrightarrow{AC}$ and $\overrightarrow{AB}$.

Solution: Complete the triangle into a parallelogram by attaching $\overrightarrow{AC}$ to point B. By assigning a mass of 1 to the vertices A,B, and C, and calling A_1 the midpoint of $\overrightarrow{BC}$, we get that $|AG| : |GA_1| = 2 : 1$, and so if we look at the diagonal of the parallelogram going through A, we have that $\overrightarrow{AG}$ is $\frac{1}{3}$ the length, which is the vector $\overrightarrow{AB} + \overrightarrow{AC}$. Therefore, $\overrightarrow{AG} = \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC})$.

4. Let C be the point on a segment AB such that $|AC| : |CB| = 2 : 7$. Expess this fact using the notion of the center of mass. Express the same fact using vectors.

Solution: C is the center of mass of 7A and 2B. In terms of vectors, we have that $7\overrightarrow{AC} + 2\overrightarrow{BC} = 0$.

5. Let G and G_1 be the centers of mass of $\triangle ABC$ and $\triangle A_1B_1C_1$ respectively. Show that

$$\overrightarrow{GG_1} = \frac{1}{3}(\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1}).$$

Solution: Note that $\overrightarrow{GG_1} = \overrightarrow{GA} + \overrightarrow{AA_1} + \overrightarrow{A_1G_1} = \overrightarrow{GB} + \overrightarrow{BB_1} + \overrightarrow{B_1G_1} = \overrightarrow{GC} + \overrightarrow{CC_1} + \overrightarrow{C_1G_1}$. Therefore, $3\overrightarrow{GG_1} = (\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}) + (\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1}) + (\overrightarrow{A_1G_1} + \overrightarrow{B_1G_1} + \overrightarrow{C_1G_1})$. The first and third terms are both 0 since G and G_1 are the centers of mass of $\triangle ABC, \triangle A_1B_1C_1$, respectively. Therefore, dividing by 3, we have that $\overrightarrow{GG_1} = \frac{1}{3}(\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1})$.

Negative masses

The same definition of the center of mass works if some (or all) of the masses in the system are negative.

Definition.

Let m_1A_1 and m_2A_2 be two point masses. Assuming that $m_1 + m_2 \neq 0$, the center of mass of this system $Z = Z(m_1A_1, m_2A_2)$ lies on the line A_1A_2 so that

$$|m_1| \cdot d_1 \;=\; |m_2| \cdot d_2,$$

where $d_1 = |\overrightarrow{ZA_1}|$ and $d_2 = |\overrightarrow{ZA_2}|$.

The center of mass Z lies between A_1 and A_2 if and only if the masses have the same sign (i.e., both are positive or both are negative).

1. Show that if $ABCD$ is a parallelogram, then $mD = Z(mA,\ (-m)B,\ mC)$.

   Solution: Let O be the midpoint of AC, then we have that $Z = Z(mA, (-m)B, mC) = Z(2mO, (-m)B)$. This must lie on the line going through OB and satisfy $2m|OZ| = m|BZ|$. So we are looking for a point outside of the segment (since the masses have opposite signs) such that $2|OZ| = |BZ|$. D is exactly this point, since O is the midpoint of BD. Moreover, this has mass $2m + (-m) = m$.

2. The base of a pyramid $SABCD$ is a parallelogram $ABCD$. A plane α intersects the sides SA, SB, SC and SD at points A_1, B_1, C_1, D_1 respectively. Given that $|SA_1| = \frac{1}{3}|SA|$, $|SB_1| = \frac{1}{5}|SB|$ and $|SC_1| = \frac{1}{4}|SC|$, find the ratio $|SD_1|/|SD|$ (use the previous problem).

   Solution: Place masses 1A, 2S, (-1)B, (-4)S, 1C, 3S. Then the centers of mass of the pairs is A_1, B_1, C_1, and so their center of mass is the center of mass of the system and lies on the place α. If instead we take the center of mass of 1A, (-1)B, and 1C, then by the previous problem we get 1D. Taking the center of mass of 2S, (-4)S, and 3S then we get 1S. Therefore, the center of mass of the system lies on SD and also on α, and so it is the point D_1. Therefore, $|SD_1|/|SD| = 1/2$.

3. Solve problem 1 from page 1 using masses of different signs. Which solution do you prefer?

Barycentric coordinates

- Let M be a point inside of $\triangle ABC$. One can find masses m_1, m_2, m_3 so that $M = Z(m_1A,\ m_2B,\ m_3C)$;
- The masses m_1, m_2, m_3 are defined up to a constant factor $k \neq 0$: $Z(km_1A,\ km_2B,\ km_3C) = Z(m_1A,\ m_2B,\ m_3C)$;
- Define k so that the sum of masses equals to 1.

Definition of barycentric coordinates

For any point M inside of $\triangle ABC$ there are positive numbers μ_1, μ_2, μ_3 so that

- $\mu_1 + \mu_2 + \mu_3 = 1$;
- $M = Z(\mu_1 A,\ \mu_2 B,\ \mu_3 C)$.

These numbers are called the ***barycentric coordinates*** *(or,* ***B-coordinates)*** ***of*** *M with respect to $\triangle ABC$.*

Warm-up problems:

1. Find the barycentric coordinates of each of the vertices.

   Solution: A = (1,0,0), B = (0,1,0), C = (0,0,1)

2. Find the point with barycentric coordinates with respect to $\triangle ABC$ are equal to $(1/2,\ 1/2,\ 0)$.

   Solution: It is the midpoint of the side AB.

3. Find the barycentric coordinates of the point of intersection of medians.

   Solution: The point is the center of mass of the triangle with equal masses on the vertices, so $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4. Let μ_1, μ_2, μ_3 be the barycentric coordinate of a point M with respect to $\triangle ABC$. Prove the following statements:

   (a) M lies on the line through points A and B if and only if $\mu_3 = 0$;

   Solution: If M lies on the line through A and B, then $|AM| : |MB| = k$, and we have that this corresponds to putting masses kA, 1B, and 0C (up to some signs), so we have that the barycentric coordinates are $(\frac{k}{k+1}, \frac{1}{k+1}, 0)$, so $\mu_3 = 0$.
   If $\mu_3 = 0$, then have that $\mu_1 \overrightarrow{AM} + \mu_2 \overrightarrow{BM} = 0$, meaning that A, M, and B are collinear.

   (b) M lies on the segment AB if and only if $\mu_1, \mu_2 > 0$ and $\mu_3 = 0$.

   Solution: By the previous problem, it suffices to assume M is on the line through A and B and $\mu_3 = 0$.
   Now if $\mu_1, \mu_2 > 0$ then we have that M is the center of mass of two masses at A and B of the same sign, so therefore M must be on the segment AB.
   If M lies on the segment AB, then again we have that μ_1, μ_2 have the same sign, and since they add up to 1, they must both be positive.

5. Draw a triangle $\triangle ABC$ and mark the following points with given barycentric coordinates: $M = (1/2,\ 1/4,\ 1/4)$, $N = (-1,\ 2,\ 0)$.

Solution: A triangle is not drawn but we describe the points. The point M is the center of mass of a triangle with masses 2A, 1B, 1C. Therefore, M is the midpoint of the median going through A.
N is the center of mass of (-1)A and 2B, so it is on the line going through A and B. N is the point outside the line segment AB, on the side of B, such that $|NB| = |AB|$. In other words, B is the midpoint of the segment NA.

6. Let $M \in BC$ be such that $|BM| = \frac{1}{3}|BC|$. Let $N \in AB$ be such that $|AN| = \frac{1}{3}|AB|$. Find the barycentric coordinates of the point of intersection of AM and CN.

Solution: We have that the point of intersection is the center of mass of 2B, 1C, and 1A. Therefore the barycentric coordinates are $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

Problems:

1. Show that the numbers μ_1, μ_2, μ_3 satisfying conditions in the definition of barycentric coordinates exist and are unique for any point M on the plane. (*Hint:* Use the fact that M is the center of mass of A, B, C if and only if for any point O we have

$$\overrightarrow{OM} = \mu_1\overrightarrow{OA} + \mu_2\overrightarrow{OB} + \mu_3\overrightarrow{OC}.$$

Choose point O in a smart way to prove the statement).

Solution: First we prove existence. It suffices to find masses for A, B, and C to make M the center of mass, because then you just divide by the total mass to get the barycentric coordinates. If M is on the line through B and C, then we have already seen that we can find barycentric coordinates for M. Otherwise, we can place a mass 1A and then solve for m_1B, m_2C. Then we are looking to solve the equation $0 = \overrightarrow{MA} + m_1\overrightarrow{MB} + m_2\overrightarrow{MC}$, or simply $\overrightarrow{AM} = m_1\overrightarrow{MB} + m_2\overrightarrow{MC}$. Writing the vectors in planar coordinates makes this a 2 by 2 system of linear equations which we can solve because the triangle $\triangle ABC$ is non degenerate and M is not on the line through A and B. Then the barycentric coordinates of M are $\frac{1}{1+m_1+m_2}(1, m_1, m_2)$.
To prove uniqueness, we use the hint. If there were another set of barycentric coordinates (μ_1', μ_2', μ_3') then choosing $O = A$, they would also satisfy $\mu_1'\overrightarrow{AM} = \mu_2'\overrightarrow{AB} + \mu_3'\overrightarrow{AC}$. If M is on the line through B and C, we have already seen that the barycentric coordinates are unique. If M is not on that line, then we can divide through by μ_1', giving us $AM = \frac{1}{\mu_1'}(\mu_2'\overrightarrow{AB} + \mu_3'\overrightarrow{AC})$. But the first set of barycentric coordinates also satisfies this, so we have that $\frac{1}{\mu_1'}(\mu_2'\overrightarrow{AB} + \mu_3'\overrightarrow{AC}) = \frac{1}{\mu_1}(\mu_2\overrightarrow{AB} + \mu_3\overrightarrow{AC})$. This gives us that $\mu_1\mu_2' - \mu_1'\mu_2 = 0$ and $\mu_1\mu_3' - \mu_1'\mu_3 = 0$. Using this, together with the fact that each set must sum to 1, gives us that they are in fact the same.

2. Based on the previous problem, describe a (graphical) way to determine the barycentric coordinates of any point M with respect to a given triangle $\triangle ABC$.

Solution: If M lies on one of the sides, then we have already seen how to find the barycentric coordinates. If it doesn't, then let K be the point where the line through A and M intersects the line through B and C. Place masses m_2B, m_3C such that K is their center of mass. Then place a mass m_1A such that M is the center of mass of $m_1A, (m_2 + m_3)K$. Then our barycentric coordinates for M are simply $(\frac{m_1}{m_1+m_2+m_3}, \frac{m_2}{m_1+m_2+m_3}, \frac{m_3}{m_1+m_2+m_3})$.

3. Find the barycentric coordinates of the point of intersection of altitudes of acute triangle $\triangle ABC$ given that its sides have lengths a, b, c and the angles are equal to α, β, γ.

Solution: Using trigonometry with the altitudes, we can find the lengths of all of the line segments on the perimeter. Making these into ratios, we get the following system of masses: $(a\cos\beta\cos\gamma)A, (b\cos\alpha\cos\gamma)B, (c\cos\alpha\cos\beta)C$. Then if we divide by the sum of the masses, $s = a\cos\beta\cos\gamma + b\cos\alpha\cos\gamma + c\cos\alpha\cos\beta$, we get the barycentric coordinates $(\frac{a\cos\beta\cos\gamma}{s}, \frac{b\cos\alpha\cos\gamma}{s}, \frac{c\cos\alpha\cos\beta}{s})$.

4. Let (m_1, m_2, m_3) and (n_1, n_2, n_3) be the barycentric coordinates of points M and N respectively. Find the barycentric coordinates of the midpoint of the segment MN.

Solution: The midpoint of the segment MN is the center of mass of 1M and 1N, which are each the center of mass of their respective triangles with masses equal to barycentric coordinates. Therefore, the midpoint is the center of mass of $(m_1+n_1)A, (m_2+n_2)B, (m_3+n_3)C$. It follows that the barycentric coordinates are $(\frac{m_1+n_1}{2}, \frac{m_2+n_2}{2}, \frac{m_3+n_3}{2})$.

5. (Barycentric coordinates as areas): Let P be a point inside of $\triangle A_1A_2A_3$. Let S, S_1, S_2, S_3 be the areas of triangles $\triangle A_1A_2A_3$, $\triangle PA_2A_3$, $\triangle PA_1A_3$ and $\triangle PA_1A_2$ respectively. Then show that the barycentric coordinates of P are:

$$\mu_1 = \frac{S_1}{S}, \quad \mu_2 = \frac{S_2}{S}, \quad \mu_3 = \frac{S_3}{S}.$$

(In other words, if $\triangle ABC$ has unit area, P is the center of mass of three masses positioned at the vertices where the masses are chosen proportinately to indicated areas).

Solution: Since barycentric coordinates are unique (Exercise 1), it suffices to show that $\frac{S_1}{S} = \mu_1$. Then by relabeling the vertices (or the same proof) the statement will hold for μ_2 and μ_3.
Let A_2A_3 be the base for both $\triangle A_1A_2A_3$ and $\triangle PA_2A_3$. Also, let B be the center of mass of μ_2A_2, μ_3A_3. Then we have that P lies on the line A_1B. Specifically, P is the center of mass of μ_1A_1 and $(\mu_2 + \mu_3)B$. By similar triangles, the ratio of the heights of the smaller triangle to the bigger one is the same as the ratio $|PB| : |A_1B|$. But this ratio is simply $\mu_1 : 1$. Therefore, we have that $\frac{S_1}{S} = \mu_1$.

6. Find the barycentric coordinates of the center of inscribed circle of $\triangle ABC$. (*Hint:* use interpretation of barycentric coordinates in terms of areas; see the previous problem).

   Solution: We compute the areas S_1, S_2, S_3. Drawing these triangles, we see that the radius of the circle, r, is the height of all three. The bases are the sides of the triangle, since it is perpendicular to the radius. Therefore, $2S_1 = r|BC|, 2S_2 = r|AC|, 2S_3 = |AB|$. Let $P = |AB| + |BC| + |AC|$ be the perimeter of the triangle. Since the barycentric coordinates are just the normalized (sum to one) coordinates of any scalar multiple, we have they are given by $(\frac{|BC|}{P}, \frac{|AC|}{P}, \frac{|AB|}{P})$.

7. Let B_1, B_2, B_3 be points on the sides of triangle $\triangle A_1A_2A_3$ such that

$$\frac{|A_2B_1|}{|A_2A_3|} = \frac{|A_3B_2|}{|A_3A_1|} = \frac{|A_1B_3|}{|A_1A_2|} = \frac{1}{4}.$$

   Let S be the area of triangle ABC. Find the area of the triangle PQR bounded by the lines A_1B_1, A_2B_2, A_3B_3.

   Solution: Let $P = A_1B_1 \cap A_3B_3$, $Q = A_1B_1 \cap A_2B_2$, and $R = A_2B_2 \cap A_3B_3$. Then the area of the triangle PQR is equal to $S - Area(PA_1A_3) - Area(QA_1A_2) - Area(RA_2A_3)$. With point mass geometry, we have that the barycentric coordinates for P are $(\frac{9}{13}, \frac{3}{13}, \frac{1}{13})$, those for Q are $(\frac{1}{13}, \frac{9}{13}, \frac{3}{13})$, and those for R are $(\frac{3}{13}, \frac{1}{13}, \frac{9}{13})$. Then we have, by exercise 5, that $Area(PA_1A_3) = S\mu_{P,2} = S\frac{3}{13}$, $Area(QA_1A_2) = S\mu_{Q,3} = S\frac{3}{13}$, and $Area(RA_2A_3) = S\mu_{R,1} = S\frac{3}{13}$. Therefore, the area of PQR is $S(1 - \frac{9}{13}) = \frac{4}{13}S$.

The problems are taken from:

M. Bulk, V. Boltyanskij, "Geometry of Mass", 1987 (in Russian)