The Geometry of Point Masses

April 2, 2020

Consider two point masses m_1 and m_2 and their center of mass (the point of balance). Let d_1 and d_2 be the distances from the point masses to the center of mass.

Recall the *Archemes' principle of levers:*

The product of the mass and the distance to the center of mass is the same for both point masses:

$$m_1 \cdot d_1 = m_2 \cdot d_2.$$

This property uniquely defines the center of mass for two point masses.

Here are the properties of the center of mass of a system of any finite number of points:

1. Any finite set of point masses has a unique center of mass.
2. For two point masses, their center of mass lies on the segment joining the points and dividing the segment in the ratio which is inverse proportional to the corresponding masses.
3. The position of the center of mass of a system of point masses does not change if we replace several point masses from the system with their total mass positioned at the center of mass of this subsystem.

Let mP denote the point mass m positioned at point P. Then:

1. $m_1P_1 = m_2P_2$ iff $m_1 = m_2$ and $P_1 = P_2$.
2. $m_1P_1 + m_2P_2 = (m_1 + m_2)P$, where P is the point on the segment P_1P_2 such that $|P_1P| \cdot m_1 = |P_2P| \cdot m_2$. In other words, $|PP_1| : |PP_2| = m_2 : m_1$.

The point mass mP, where $m = m_1 + m_2$, is called the *center of mass* of the point masses m_1P_1 and m_2P_2.

Warm-up problems:

1. Let Z be the center of mass of two point masses, $3P$ and $5Q$. Find m and the ratio $|PZ| : |ZQ|$.

   Solution: $m = 8, |PZ| : |ZQ| = 5 : 3$.

2. Let Z be the center of mass of two point masses, $7A$ and mP. Find m and the ratio $|AZ| : |ZP|$ given that $7A + mP = 10Z$.

   Solution: $m = 3, |AZ| : |ZP| = 3 : 7$.

Solve the following geometry problems using point masses:

1. Let AD be the median bisecting the side BC in $\triangle ABC$. Let $Z \in AD$ be the point on AD such that $|AZ| = |ZD|$. Find the ratio in which the line going through B and Z divides the side AC.

   Solution: Put masses 2A, 1B, 1C so that 4Z is the center of mass of the system. Then if we first take the center of mass of 2A and 1C, say 3P, we have that 4Z is the center of mass of 3P and 1B. Therefore, we have that the line going through B and Z divides the side AC at point P in a ratio of 1 : 2.

2. Let M be the point on the side AC of $\triangle ABC$ such that $|AM| = \frac{1}{3}|AC|$. Let N be the point on extension of the side BC beyond point B so that $|BN| = |NC|$. Let P be point of intersection of AB with MN. Find the ratios $|AP| : |PB|$ and $|NP| : |PM|$.

   Solution: Put masses 2A, 1C, 1N so that 2B is the center of mass of 1N and 1C, and 3M is the center of mass of 2A and 1C. Then the point P is the center of mass of the system. We have that $|AP| : |PB| = 1 : 1$ and $|NP| : |PM| = 3 : 1$.

3. Let $ABCD$ be a convex quadrilateral. Let K, L, M, N be the midpoints of the sides AB, BC, CD and DA respectively. Let O be the point of intersection of KM and LN.

   (a) Show that O is the midpoint of both KM and LN.

   Solution: Put a mass of 1 at each vertex of the quadrilateral ABCD. Then we have that 2K, 2L, 2M, 2N are the center of masses of their respective sides. The center of mass of the system is O, since it lies on both KM and LN (obtained by combining opposite sides first). Therefore, O is the center of mass of 2K and 2M as well as of 2L and 2N. Therefore, O bisects both KM and LN.

   (b) Show that O is also the midpoint of the segment connecting the midpoints of diagonals of the quadrilateral.

   Solution: If we combine opposite vertices first, we have that O is the center of mass of the midpoints of the diagonals. Moreover, the midpoints of the diagonals will have equal mass, so O is the midpoint of the segment connecting them.

4. Let D be the midpoint of BC in $\triangle ABC$. Let E be the point on AC such that $|AE| : |EC| = 1 : 3$. Let $K = BE \cap AD$. Find the ratios $|AK| : |KD|$ and $|BK| : |KE|$.

   Solution: Put masses 3A, 1B, 1C. Then it follows that 2D is the center of mass of 1B and 1C, and 4E is the center of mass of 3A and 1C. Then K is the center of mass of the system. Specifically, it is the center of mass of 3A and 2D, giving $|AK| : |KD| = 2 : 3$. It is also the center of mass of 1B and 4E, giving $|BK| : |KE| = 4 : 1$.

5. A line goes through the vertex A of triangle $\triangle ABC$ and the midpoint L of the median BB_1. Let K be the point of intersection of AL and the median CC_1. Find the ratio $|CK| : |KC_1|$.

   Hint: This problem can not be solved in one step. Follow the plan below to solve the problem:

   (a) First, place point masses at the vertices of $\triangle ABC$ in such a way that L is the center of mass. Let $P = AL \cap BC$. Use the point masses to find the ratio $|BP| : |PC|$.

   Solution: L is the center of mass of the system 1A, 2B, 1C. Then we have that P is the center of mass of 2B and 1C, so $|BP| : |PC| = 1 : 2$.

   (b) Now place another set of point masses at the vertices of $\triangle ABC$ in such a way that K is the center of mass. Now find the ratio $|CK| : |KC_1|$.

   Solution: Put masses 2A, 2B, 1C so that $4C_1$ is the center of mass of 2A and 2B and 3P is the center of mass of 2B and 1C. Then we have that the center of mass of the system is K. Specifically, K is the center of mass of 1C and $4C_1$, so $|CK| : |KC_1| = 4 : 1$.

6. Let $ABCD$ be a parallelogram. Let l be the line going through D and crossing the segment AB at the point K in such a way that $|AK| = \frac{1}{n}|AB|$ for $n \in \mathbb{N}$, $n \geq 2$. In what ratio does this line divide the diagonal AC?

   Solution: Put masses (n-1)A, 1B, 1C so that nK is the center of mass of (n-1)A and 1B. Therefore, the center of mass of the system, call it P, lies on DK. It also lies on AO, where O is the midpoint of the diagonal BD. We have that $|AP| : |PO| = 2 : n - 1$. But we also know that $|AO| = |OC|$, since the diagonals of a parallelogram bisect each other. Therefore, the line l divides the diagonal AC is a ratio $2 : (n - 1) + (n + 1) = 2 : 2n = 1 : n$.

7. (Varignon's Theorem) If the midpoints of consecutive sides of a quadrilateral are connected, the resulting quadrilateral is a parallelogram.

   (a) Solve the problem using point masses.

Solution: It suffices to show that the diagonals of the resulting quadrilatel bisect each other. From there, it follows that it must be a parallelogram. If we start with a quadrilateral $ABCD$ and put a mass of 1 at each vertex, then the midpoints M, N, P, Q are the centers of mass of each corresponding side and have equal weight. The center of mass of the system can be obtained by first taking the centers of mass of 1A, 1B and 1C, 1D giving 2M and 2P, respectively. Then taking the center of mass of 2M and 2P, we get that it bisects MP. We can also find the center of mass of the system by first combining 1A,1D and 1B, 1C, giving 2Q and 2N, respectively. Then taking the center of mass of those, we see that it bisects NQ. Therefore, the center of mass is the intersection of the diagonals and they bisect each other.

(b) Solve the problem using geometry only.

8. Let $ABCD$ be a quadrilateral such that a circle can be insribed into it. Let $M \in AB$, $N \in BC$, $P \in CD$ and $Q \in AD$ be the points of tangency of the inscribed circle. Suppose that $|AM| = a$, $|BN| = b$, $|CP| = c$ and $|DQ| = d$. Let $Z = MP \cap NQ$. Find the ratios $|MZ| : |ZP|$ and $|QZ| : |ZN|$.

Solution: Solve for all the lengths of the segments using the fact that the lengths from a point to two tangencies are equal. Put masses $1A$, $\frac{a}{b}B$, $\frac{a}{c}C$, $\frac{a}{d}D$ so that the tangencies are the center of masses of the sides. Then we get masses $(1+\frac{a}{b})M, a(\frac{1}{b}+\frac{1}{c})N, a(\frac{1}{c}+\frac{1}{d})P, a(\frac{1}{d}+1)Q$, to simplify, we can rescale by $\frac{1}{a}$, giving $(\frac{1}{a}+\frac{1}{b})M, (\frac{1}{b}+\frac{1}{c})N, (\frac{1}{c}+\frac{1}{d})P, (\frac{1}{d}+\frac{1}{a})Q$ Then $|MZ| : |ZP| = (\frac{1}{c}+\frac{1}{d}) : (\frac{1}{a}+\frac{1}{b})$ and $|QZ| : |ZN| = (\frac{1}{b}+\frac{1}{c}) : (\frac{1}{d}+\frac{1}{a})$.

9. Let M and N be the points on the sides AC and BC of $\triangle ABC$ respectively, so that $|AM| : |MC| = 3 : 1$ and $|BP| : |PC| = 1 : 2$. Let $Q = AP \cap BM$. Given that area of $\triangle BPQ$ is equal to 1 in^2, find the area of $\triangle ABC$.

Solution: Viewing BP and BC as the base for each triangle, we can use similar triangles to show that the ratio of the heights is the same as the ratio $|PQ| : |PA|$. Put masses $1A, 6B, 3C$, which makes Q the center of mass. Then we have that $|AQ| : |QP| = 9 : 1$, so $|PQ| : |PA| = 1 : 10$. Therefore, the height is scaled by 10 and the base is scaled by 3, so the overall area must be scaled by 30, giving 30 in^2.

10. Let $PABC$ be a triangular pyramid with apex P such that

- $\triangle ABC$ is equilateral;
- the line going through the apex P and the center M of $\triangle ABC$ is the altitude of the pyramid (i.e., PM is perpendicular to the plane of $\triangle ABC$).

A plane α intersects the pyramid in such a way that it divides the sides PA, PB and PC in the ratios $2:3$, $3:2$, and $4:1$ respectively. Find the ratio in which the plane divides the altitude PM of the pyramid.

Solution: Put masses $12A, 18P, 12B, 8P, 12C, 3P$ so that the intersection points of each line with the plane is the center of mass of the two points. Then the center of mass of the system lies on the plane (since it is the center of mass of the intersection points) but it is also on the line PM, since M is the center of mass of the base triangle. Adding the masses, we have $36M, 29P$, giving a ratio $36:29$ for the plane to divide the altitude PM.