Diophantine chord technique

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Definition: An *integral point* is a point whose all coordinates are integers. Similarly for *rational points*.

Question: We want to study when curves related to rational algebraic expressions contain integral and rational points.

Question: In particular, when does the curve $x^2 + y^2 - z^2 = 0$ hit rational points in $\mathbb{R}^3$?

1. Find all integral points in the line given by $y = \frac{5x}{17} + \frac{1}{17}$.

   Hint: Take the GCD of 5 and 17.

2. Find all integer lattice point in the line L given by $y = \frac{x}{\sqrt{2}} + \sqrt{3}$

   Hint: Use that $\sqrt{6}$ (or $\sqrt{2}$) is not rational.

3. Find all rational solutions to the equation $x^5 - 9x^4 - 5x^3 + 45x^2 + 4x - 36 = 0$

Hint: We prove the following theorem: Let $f(t) = a_n t^n + \cdots + a_0$. If $\frac{p}{q} \in \mathbb{Q}$, with $\gcd(p, q) = 1$ is a root of f, then p divides a_0 and q divides a_n.

4. Show that $\sqrt{2}$ is irrational via $x^2 - 2 = 0$.

Hint: Previous theorem

5. Show that $p^{\frac{1}{n}}$ is irrational for all p prime and $n \in \mathbb{Z}$.

Hint: Previous theorem

6. Find all rational points on $x^2 + y^2 = 1$.

Solution: Let L be the line passing through $(0, 1)$ and another rational point on the circle. This implies that the slope is rational. The line is of the form $y = mx + 1$. Substituting gives: $x = \frac{-2m}{1+m^2}$ and $y = \frac{1-m^2}{1+m^2}$. Together with $(0, 1)$ these are all rational points. Can you find all Pythagorean triples $((x, y, z) \in \mathbb{Z}^3$ with $x^2 + y^2 = z^2)$?

7. Find all rational points on $x^2 - 7y^2 = 1$.

Solution: $P = (1, 0)$, $y = m(x - 1)$. Then the equation reduces to $x = 1$ or $x = \frac{7m^2+1}{7m^2-1}$.

8. Find all rational points on $x^2 - 7y^2 = 3$.

Solution: Mod 7, we see there are no solutions.

9. Find all rational points on $y^2 = x^3 + 1$.

Hint: $P = (-1, 0)$, $y = m(x + 1)$.

10. Find all rational points of the system $x^2y^2 + x^2 = u^2$ and $x^2y^2 + y^2 = v^2$.

Solution: Diophantus finds one rational solution, namely $(x, y) = (3/4, 7/24)$. Let us find all the integral solutions first. From the first equation we see that $x^2(y^2 + 1) = u^2$. Therefore, either $x = u = 0$ or $y^2 + 1$ itself is a square. Since the only two consecutive squares are 0 and 1, it follows that $(x, y, u, v) = (n, 0, \pm n, 0)$ and $(0, m, 0, \pm m)$, for some integers m, n, are the only integral solutions. Now, let us find the rational solutions. As before, $x = u = 0$ or $y^2 + 1$ is a square. Thus, there is $t \in \mathbb{Q}$, with $t \neq \pm 1$, such that $y = \frac{2t}{1-t^2}$ (this follows from parametrizing $y^2 + 1 = w^2$). Now the second equation says $y^2(x^2+1) = v^2$, so either $y = 0$ or x^2+1 is a square. We similarly conclude that $x = \frac{2s}{1-s^2}$ for some $s \in \mathbb{Q}$ with $s \neq \pm 1$. Hence, the rational solutions of the problem are given by

$$(x, y) = (\frac{2s}{1-s^2}, \frac{2t}{1-t^2}),$$

and there is a solution for each $s, t \in \mathbb{Q}$ other than ± 1.

11. Generalize these results as much as you can.

(Homework)