Applications of Expected Value

Jacob Zhang, Shend Zhjeqi

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1 Definitions and Theorems

Definition 1.1. The **expected value** $\mathbb{E}[X]$ of a discrete random variable X defined on a probability space Ω is defined as

$$\mathbb{E}[X] = \sum_{\substack{a \in \mathbb{R} \\ \mathbb{P}(X=a)>0}} a\mathbb{P}(X = a)$$

Theorem 1 (Linearity of Expectation)**.** *If X and Y are random variables, then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.*

Theorem 2. *If X and Y are independent random variables, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*

2 Problems

1. You're trapped in a casino with \$1 in your pocket. It will cost \$2020 to bribe the guard to let you escape. Fortunately, you have access to a slot machine which costs \$1 to play and gives you back \$2 with probability $\frac{2}{3}$. Show that you can escape with probability strictly greater than $\frac{1}{2}$.

2. (Putnam 2014) Suppose X is a random variable that takes on only nonnegative integer values, with $\mathbb{E}[X] = 1$, $\mathbb{E}[X^2] = 2$, and $\mathbb{E}[X^3] = 5$. Determine the smallest possible value of the probability of the event $X = 0$.

3. (Week 4, IMC 2002) Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

4. (Week 1) Prove the Principle of Inclusion-Exclusion: For finite sets $A_1, A_2, \ldots A_n$,

$$|\bigcup_i A_i| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots$$

5. Solving permutation problems with indicator variables:

   (a) (Fixed points) A group of 6 people playing Secret Santa write their names on cards and put the cards in a hat, then randomly draw out names one by one without replacement. What is the expected number of people who get their own name back?

   (b) (Inversions) Bob generates a random permutation σ of the numbers 1 through n. Then every second, he chooses a pair of neighboring numbers $\sigma(i)$ and $\sigma(i + 1)$ such that $\sigma(i) > \sigma(i + 1)$, choosing uniformly at random if there are multiple pairs, and swaps them, replacing σ with the permutation σ' which is the same except $\sigma'(i) = \sigma(i + 1)$ and $\sigma'(i + 1) = \sigma(i)$. Bob stops when there are no more such pairs of neighboring numbers. What is the expected number of seconds before Bob stops?

(c) (Visibility) A city planner randomly arranges n buildings, of height 1 through n, in a row from west to east along a street. A building is called *visible from the west* if it is taller than every building west of it. What is the expected number of buildings visible from the west?

6. Let $X_1, X_2, \ldots, X_n$ be independent uniform random variables on $[0, 1]$.

   (a) What is the expected value of $\min(X_1, X_2, \ldots X_n)$?

   (b) What is the expected value of the kth smallest among $X_1, X_2, \ldots X_n$?

   (c) What is the expected value of the pth power of the kth smallest among $X_1, X_2, \ldots X_n$?

   (d) To use a particular sword in Dungeons and Dragons, you roll 3 D20s (dice with 20 sides labeled from 1 to 20) and then do damage equal to the highest roll. Approximately what is the expected amount of damage this sword does? Can you be more exact?

3 Solutions

1. Let X_n be the random variable describing how much money you have after n uses of the slot machine. Note that $\mathbb{E}[2^{-X_{n+1}}] = 2^{-X_n}$ (i.e. 2^{X_n} is a martingale). We argue that the player reaches one of the two end states $X_n = 2020$ and $X_n = 0$ with probability 1. Let p be the probability of winning. Since 2^{-X_n} is invariant, we have $2^{-1} = p2^{-2020} + (1-p)1$, so $p = \frac{1/2}{1-2^{-2020}} > \frac{1}{2}$.

2. Let $Y = (X-1)(X-2)(X-3)$. Compute $\mathbb{E}[Y] = \mathbb{E}[(X-1)(X-2)(X-3)] = \mathbb{E}[X^3] - 6\mathbb{E}[X^2] + 11\mathbb{E}[X] - 6 = -2$. The only time $Y < 0$ is when $X = 0$, in which case $Y = -6$. Therefore, $\mathbb{E}[Y] \geq -6\mathbb{P}(X = 0)$, and so $\mathbb{P}(X = 0) \geq \frac{1}{3}$. As suggested by the calculation above, we should try to achieve equality with a distribution supported on $\{0, 1, 2, 3\}$. Suppose this is the case; then note that $6\mathbb{P}(X = 3) = \mathbb{E}[X(X-1)(X-2)] = \mathbb{E}[X^3] - 3\mathbb{E}[X^2] + 2\mathbb{E}[X] = 1$. Similar logic gives $\mathbb{P}(X = 0) = \frac{1}{3}$, $\mathbb{P}(X = 1) = \frac{1}{2}$,$\mathbb{P}(X = 2) = 0$,$\mathbb{P}(X = 3) = \frac{1}{6}$.

3. Let X be a uniformly randomly chosen student and let $X_i = 1$ if they missed problem i, 0 otherwise. Let Y be iid copy of X. Compute $\mathbb{E}[\sum X_i Y_i] = 6\mathbb{E}[X_1]^2 \leq 6 \cdot (2/5)^2 = \frac{24}{25} < 1$. Therefore some pair of students (X, Y) have $\sum X_i Y_i = 0$, i.e. together solved every problem. Note that we didn't enforce $X \neq Y$, but it doesn't matter, because if $X = Y$ that means student X solved every problem on their own.

4. Take a uniform random distribution over the union and put in indicator variables for membership in each subset.

5. All are fairly standard.

6. Subproblems:
   (a) $\frac{1}{n+1}$, add an additional X_{n+1}.
   (b) $\frac{k}{n+1}$, add an additional X_{n+1}.
   (c) $\frac{k(k+1)\cdots(k+p-1)}{(n+1)(n+2)\cdots(n+p)}$, add an additional $X_{n+1}, \ldots, X_{n+p}$.
   (d) Estimate: 15.5, exact: a bit harder