Some useful results

Make sure that you know the following results!

Lemma: Let $AB$ be a chord on a circle with center $O$. Let $P$ be a point in the same side as $O$ with respect to $AB$. Then, $\angle AOB = 2 \cdot \angle APB$.

Lemma: A convex quadrilateral $ABCD$ is cyclic if and only if $\angle ABC + \angle ADC = 180$. Equivalently, $\angle ABD = \angle ACD$.

Exercise: There is a nontrivial symmetry in the above lemma. What is it? Also, what happens if the quadrilateral is not convex?

Lemma: Let $ABC$ be a triangle with incenter $I$, $A$-excenter $I_A$, and denote by $L$ the midpoint of arc $BC$. Show that $L$ is the center of a circle through $I, I_A, B, C$.

Tangent criterion: Suppose $ABC$ is inscribed in a circle with center $O$. Let $P$ be a point in the plane. Then the following are equivalent.

(i) $PA$ is tangent to the circumcircle of $ABC$
(ii) $OA$ is perpendicular to $AP$
(iii) $\angle PAB = \angle ACB$

Ceva’s theorem: Let $AX, BY, CZ$ be cevians of a triangle $ABC$. They concur if and only if $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

Problems

Problem 1: Triangle $ABC$ has incenter $I$. Consider the triangle whose vertices are the circumcenters of $IAB, IBC, ICA$. Show that its circumcenter coincides with the circumcenter of $ABC$.

Problem 2:(Simson line) Let $ABC$ be a triangle and $P$ be any point on its circumcircle. Let $X, Y, Z$ be the feet of the perpendiculars from $P$ onto lines $BC, CA, AB$. Prove that points $X, Y, Z$ are collinear.

Problem 3:(BAMO) In an acute triangle $ABC$ let $K, L, M$ be the midpoints of sides $AB, BC, CA$, respectively. From each of $K, L, M$ drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from $K$ to sides $BC$ and $CA$, etc. The resulting 6 perpendiculars intersect at points $Q, S, T$ as in the figure to form a hexagon $KQLSMT$ inside triangle $ABC$. Prove that the area of this hexagon $KQLSMT$ is half of the area of the original triangle $ABC$. 

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