Winter Quarter Week 6: Geometry Handout

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Some useful results

Make sure that you know the following results!

Lemma: Let AB be a chord on a circle with center O. Let P be a point in the same side as O with respect to AB. Then, $\angle AOB = 2 \cdot \angle APB$.

Lemma: A convex quadrilateral $ABCD$ is cyclic if and only if $\angle ABC + \angle ADC = 180$. Equivalently, $\angle ABD = \angle ACD$.
Exercise: There is a nontrivial symmetry in the above lemma. What is it? Also, what happens if the quadrilateral is not convex?

Lemma: Let ABC be a triangle with incenter I, A-excenter I_A, and denote by L the midpoint of arc BC. Show that L is the center of a circle through I, I_A, B, C.

Tangent criterion: Suppose ABC is inscribed in a circle with center O. Let P be a point in the plane. Then the following are equivalent.
(i) PA is tangent to the circumcircle of ABC
(ii) OA is perpendicular to AP
(iii) $\angle PAB = \angle ACB$

Ceva's theorem: Let AX, BY, CZ be cevians of a triangle ABC. They concur if and only if $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$.

Problems

Problem 1: Triangle ABC has incenter I. Consider the triangle whose vertices are the circumcenters of IAB, IBC, ICA. Show that its circumcenter coincides with the circumcenter of ABC.

Problem 2:(Simson line) Let ABC be a triangle and P be any point on its circumcircle. Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y , Z are collinear.

Problem 3:(BAMO) In an acute triangle ABC let K, L, and M be the midpoints of sides AB, BC, and CA, respectively. From each of K, L, and M drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from K to sides BC and CA, etc. The resulting 6 perpendiculars intersect at points Q, S, and T as in the figure to form a hexagon $KQLSMT$ inside triangle ABC. Prove that the area of this hexagon $KQLSMT$ is half of the area of the original triangle ABC.