Winter Quarter Week 5: Algebra Review

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1. (BAMO 2015 p1) Which number is larger A or B?

$$A = \frac{1}{2015}(1 + \frac{1}{2} + \cdots + \frac{1}{2015}) \quad \text{and} \quad B = \frac{1}{2016}(1 + \frac{1}{2} + \cdots + \frac{1}{2016})$$

2. (AIME I 2014 # 1) Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4$$

There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

3. (BAMO 2014 p3) Suppose that for two real numbers x and y the following equality is true:

$$(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = 1.$$

Find (with proof) the value of $x + y$.

4. (BAMO 2011 p3) Let S be a finite, nonempty set of real numbers

such that the distance between any two distinct points in S is an element of S. In other words, $|x - y|$ is in S whenever $x \neq y$ and x and y are both in S. Prove that the elements of S may be arranged in an arithmetic progression. This means that there are numbers a and d such that $S = \{a, a + d, a + 2d, a + 3d, \cdots, a + kd, \cdots\}$.

5. (BAMO 2017 p5) Call a number T persistent if the following holds: Whenever a, b, c, d are real numbers different from 0 and 1 such that

$$a + b + c + d = T,$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = T,$$

we also have

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d} = T$$

a) If T is persistent, prove that T must be equal to 2.

b) Prove that 2 is persistent.