Winter Quarter Week 3: Combinatorics

Jacob Zhang, Shend Zhjeqi

2 February 2020

1. Let r $= 2k$. By assumption, for each value $i = 1, 2, \cdots, r$, player P_{n-i} won either less than $n-r$ or more than $n+r$ games. In the first case, player P_{n-i} must have lost more than $r-i$ games against lower rated players; in the second case, player P_{n-i} must have won more than $r+i$ games against higher-rated players. Either way, P_{n-i} participated in at least $r-i+1$ upsets. By similar arguments, for each $i = 1, 2, \cdots, r$, player P_{n+i} also participated in at least $r-i+1$ upsets, and player P_n participated in at least $r+1$ upsets. Thus, altogether, we have at least $[r + (r-1) + (r-2) + \cdots + 1] + [r + (r-1) + (r-2) + \cdots + 1] + (r+1) = (r+1)^2$ participations in upsets. Every upset involved precisely two players, so this requires a total of at least $\frac{(r+1)^2}{2}$ upsets. However, $\sqrt{2k} < r+1$, so $k < \frac{(r+1)^2}{2}$. So the number of upsets is less than $\frac{(r+1)^2}{2}$, a contradiction.

2. Notice: The sum of the squares is a monovariant. This is true because

$$(a^2 + b^2) - [(\frac{a-b}{2})^2 + (\frac{a+b}{2})^2] = \frac{a^2 + b^2}{2} \geq 0.$$

   Moreover, in the first step that we apply our operation, this monovariant will strictly decrease as all the numbers are nonzero. Hence, we are done.

3. Let us assume that the contrary is true. That is, for every two students, there is some problem that neither of them solved. This prompts us to count the pairs of students with their unsolved problem. Let us consider the matrix associated to this configuration. We have six rows, each representing a problem, and 200 columns, each representing a student. In light of the above remark, we make an entry of the matrix 1 if the student corresponding to the column did not solve the problem corresponding to the row, and make the entry 0 otherwise. Let T denote the set of pairs of 1's that belong in the same row. Let us consider the cardinality of T from two different perspectives.

   <u>Counting by columns:</u> We assumed that for every two students, there was a problem that neither of them solved. Thus, for every two columns, there is at least one pair of 1's among these two columns that belong in the same row. So we can find an element of T in every pair of columns. Since there are $\binom{200}{2}$ pairs of columns, we have $|T| \geq \binom{200}{2} = 19,900$.

   <u>Counting by rows:</u> We are told that each problem was solved by at least 120 students. This means that there are at most 80 ones in each row. So each row contains at most $\binom{80}{2}$ pairs of 1's. Since there are six rows, we have $|T| \leq 6\binom{80}{2} = 18,960$. Combining the above two inequalities, we get $19,900 \leq |T| \leq 18,960$, which is clearly absurd. Therefore, our initial assumption must be false. So there must be two students such that every problem was solved by at least one of these two students.

4. Azul player wins.

   Azul starts by picking triangle $A_1A_{1010}A_{1011}$, the points are split into two sets $S = \{A_2, A_3, \ldots A_{1009}\}$ and $\overline{S} = \{A_{1012}, A_{1013}, \ldots A_{2019}\}$. It is clear that any other triangle must have all its vertices in one of these two sets. On the successive rounds Azul just mimic the previous triangle chosen by Rojo in the complementary set - the two sets are symmetric.

One can show that at the end : 1- Azul and Rojo are even on $S \cup \overline{S}$. 2- Rojo can get at most one of A_{1010}, A_{1011}. 3- A_1 is for Azul.

OR Notice that there is an odd number of vertices, so there can be no draw. Hence, as Azul performs the symmetric action that Rojo performs, Azul will have at least as many vertices as Rojo. Those two imply that Azul wins.

5. Let us write $n = 10001$. Denote by T the set of ordered triples (a, C, S), where a is a student, C a club, and S a society such that $a \in C$ and $C \in S$. We shall count $|T|$ in two different ways. Fix a student a and a society S. We have that $|T| = nk$. Now fix a club C. If C is the set of all clubs, we obtain $|T| = \sum_{C_i \in clubs} \frac{|C_i|(|C_i|-1)}{2}$ because for each club, we can pick any person and there will be precisely $\frac{|C_i|-1}{2}$ societies. But we also, this is counting the number of ways we could choose pairs of students from given group. From (i), $\sum_{C_i \in clubs} \frac{|C_i|(|C_i|-1)}{2} = \frac{n(n-1)}{2}$. Therefore $\frac{n(n-1)}{2} = nk$, i.e., $k = \frac{n-1}{2} = 5000$.