Winter Quarter Week 4: Combinatorics

Jacob Zhang, Shend Zhjeqi

2 February 2020

1. A chess tournament took place between $2n+1$ players. Every player played every other player once, with no draws. In addition, each player had a numerical rating before the tournament began, with no two players having equal ratings.

   It turns out there were exactly k games in which the lower-rated player beat the higher-rated player. Prove that there is some player who won no less than $n-\sqrt{2k}$ and no more than $n+\sqrt{2k}$ games.

2. The nonzero numbers $a_1, \cdots, a_n \in \mathbb{R}$ are written on the board. One can make a "move" by replacing two numbers a, b by $\frac{a+b}{2}$ and $\frac{a-b}{2}$. Show that after any finite number of moves one can never come back to the same list of number that we started with.

3. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

4. We have a regular polygon with 2019 vertices, and in each vertex there is a coin. Two players Azul and Rojo take turns alternately, beginning with Azul, in the following way: first, Azul chooses a triangle of three vertices of the polygon and colors its interior with blue, then Rojo selects three vertices of the polygon and colors the interior with red, so that the triangles formed in each move don't intersect internally the previous colored triangles. They continue playing until it's not possible to choose another triangle to be colored. Then, a player wins the coin of a vertex if he colored the greater quantity of triangles incident to that vertex (if the quantities of triangles colored with blue or red incident to the vertex are the same, then no one wins that coin and the coin is deleted). The player with the greater quantity of coins wins the game. Find a winning strategy for one of the players.

5. There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

   (i) Each pair of students is in exactly one club.

   (ii) For each student and each society, the student is in exactly one club of the society.

   (iii) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies. Find all possible values of k.