Winter Quarter Week 3: Combinatorics

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- Solution for P1: Let $x = 8^{2010}$. Let $S(x)$ be the sum of its digits in the decimal expansion. Then, $x - S(x)$ is divisible by 9. So, $x \equiv S(x) \pmod 9$. As $x \equiv 1 \pmod 9$, we have that $S^k(x) = 1$ for large enough k.

- Solution for P2: Notice that the sum of all elements is invariant $\pmod 2$. As the number can be 0 or 1, it is invariant with respect to the elimination process we follow.

- Solution for P3:Notice that $|a - b| \equiv a + b \pmod 2$. Thus, the sum of all the elements is invariant $\pmod 2$.

- Solution for P4: The discriminant is invariant. Thus, it is impossible.

- Solution for P5: Step 1: Start with one person x_1.

  Step 2: There is somebody connected to it, say x_2. At step k, we are going to have a complete k-graph $\{x_1, \cdots, x_k\}$. For $k \leq n$, pick $S = \{x_1, \cdots, x_k\}$. There is somebody outside S connected to all of them. Because of this, we can assume that we have a $n + 1$ complete subgraph. Now, let S' be the rest. By problem's condition there is somebody in our $n + 1$ complete graph connecting to the rest. Hence, we are done.

- Solution to P6: Let the number of such triangles be k. For each edge between two points in the set we count the number of triangles it is part of. Let the total number over all edges be T. On the one hand, for any edge AB, there are at most 4 points such that the triangles they form with A and B have the same area. This is because those points have to be the same distance from line AB, and no three of them are collinear. Thus, $T \leq 4\binom{n}{2}$ On the other hand, each triangle has 3 edges, so $T \geq 3k$. These two give the result.

- Solution 1 to P7: Count the total number of elements throughout all the subsets of $\{1, \cdots, n\}$. In one way, every number is in precisely 2^{n-1} such sets. Hence, this number is $n2^{n-1}$. On the other hand, There are precisely $\binom{n}{i}$ sets of length i, so the sum is also $\sum_{i=0}^{n} i\binom{n}{i}$.

- Solution 2 to P7: $\sum_{i=1}^{n} i\binom{n}{i} = \sum_{i=0}^{n} i\frac{n(n-1)!}{i!(n-i)!} = n\sum_{i=1}^{n} \binom{n-1}{i-1} = n2^{n-1}$

- Solution 3 to P7: Let $f(x) = (1 + x)^n = \sum_{i=0}^{n} \binom{n}{i}x^i$. Then, $f'(x) = n(1 + x)^{n-1} = \sum_{i=0}^{n} i\binom{n}{i}x^{i-1}$. Set $x = 1$ to get the result.

- Solution 4 to P7: (Using nontrivial symmetries) Use the fact that $\binom{n}{i} = \binom{n}{n-i}$. Sum the LHS-expression twice and use the expansion of 2^n with the binomial formula.

  Remark: The nice thing about the fourth solution is that it is a solution where one considers all non-trivial symmetries and combines them together to get some result. In general, this idea produces interesting results.

- Solution to P8: The first idea that might occur here would be to find $f_k(n)$, then multiply it by k, sum it up... probably resulting in a big expression. However, if we look at the required result, we see that it suggests a natural counting - the left hand side is the total number of fixed points over all permutations. Another way to obtain that is to consider that each element of $\{1, 2, 3, \cdots, n\}$ is a fixed point in $(n-1)!$ permutations, so the total is $n(n-1)! = n!$. (Note that we are counting pairs of the form (point, permutation) such that the point is a fixed point of the permutation.