Winter Quarter Week 3: Combinatorics

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• Solution for P1: Let \( x = 8^{2010} \). Let \( S(x) \) be the sum of its digits in the decimal expansion. Then, \( x - S(x) \) is divisible by 9. So, \( x \equiv S(x) \) (mod 9). As \( x \equiv 1 \) (mod 9), we have that \( S^k(x) = 1 \) for large enough \( k \).

• Solution for P2: Notice that the sum of all elements is invariant (mod 2). As the number can be 0 or 1, it is invariant with respect to the elimination process we follow.

• Solution for P3: Notice that \( |a - b| \equiv a + b \) (mod 2). Thus, the sum of all the elements is invariant (mod 2).

• Solution for P4: The discriminant is invariant. Thus, it is impossible.

• Solution for P5: Step 1: Start with one person \( x_1 \).
  Step 2: There is somebody connected to it, say \( x_2 \). At step \( k \), we are going to have a complete \( k \)-graph \( \{x_1, \ldots, x_k\} \). For \( k \leq n \), pick \( S = \{x_1, \ldots, x_k\} \). There is somebody outside \( S \) connected to all of them. Because of this, we can assume that we have a \( n + 1 \) complete subgraph. Now, let \( S' \) be the rest. By problem’s condition there is somebody in our \( n + 1 \) complete graph connecting to the rest. Hence, we are done.

• Solution to P6: Let the number of such triangles be \( k \). For each edge between two points in the set we count the number of triangles it is part of. Let the total number over all edges be \( T \). On the one hand, for any edge \( AB \), there are at most 4 points such that the triangles they form with \( A \) and \( B \) have the same area. This is because those points have to be the same distance from line \( AB \), and no three of them are collinear. Thus, \( T \leq 4 \binom{n}{2} \). On the other hand, each triangle has 3 edges, so \( T \geq 3k \). These two give the result.

• Solution 1 to P7: Count the total number of elements throughout all the subsets of \( \{1, \ldots, n\} \). In one way, every number is in precisely \( 2^{n-1} \) such sets. Hence, this number is \( n2^{n-1} \). On the other hand, there are precisely \( \binom{n}{i} \) sets of length \( i \), so the sum is also \( \sum_{i=0}^{n} i \binom{n}{i} \).

• Solution 2 to P7: \( \sum_{i=1}^{n} i \binom{n}{i} = \sum_{i=0}^{n} i \frac{n(n-1)!}{i!(n-i)!} = n \sum_{i=1}^{n} \binom{n-1}{i-1} = n2^{n-1} \)

• Solution 3 to P7: Let \( f(x) = (1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \). Then, \( f'(x) = n(1 + x)^{n-1} = \sum_{i=0}^{n} i \binom{n}{i} x^{i-1} \). Set \( x = 1 \) to get the result.

• Solution 4 to P7: (Using non-trivial symmetries) Use the fact that \( \binom{n}{i} = \binom{n}{n-i} \). Sum the LHS-expression twice and use the expansion of \( 2^n \) with the binomial formula.

Remark: The nice thing about the fourth solution is that it is a solution where one considers all non-trivial symmetries and combines them together to get some result. In general, this idea produces interesting results.
Solution to P8: The first idea that might occur here would be to find $f_k(n)$, then multiply it by $k$, sum it up... probably resulting in a big expression. However, if we look at the required result, we see that it suggests a natural counting - the left hand side is the total number of fixed points over all permutations. Another way to obtain that is to consider that each element of \{1, 2, 3, \cdots, n\} is a fixed point in $(n - 1)!$ permutations, so the total is $n(n - 1)! = n!$. (Note that we are counting pairs of the form (point, permutation) such that the point is a fixed point of the permutation.)