Winter Quarter Week 3: Combinatorics

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1. The number 8^n is written on the board. The sum of its digits is calculated, then the sum of the digits of the result is calculated, and so on, until a single digit is reached. What is this digit if $n = 2010$?

2. Write 11 numbers on the board–six zeros and five ones, in any order. Perform the following operation 10 times: cross out any two numbers, and if they were equal, write another zero on the board. If they were not equal, write a one. Determine the final number on the board, and show that it does not depend on the manner in which the numbers were chosen at each step.

3. Suppose that the positive integer n is odd. Write the numbers $\{1, 2, \cdots, 2n\}$ on the board. Choose any 2 numbers a and b, erase them, and write $|a - b|$. Determine whether the final number on the board will be odd or even, and show that it does not depend on the manner in which the numbers were chosen at each step.

4. The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$:

   (a) switch a and c, and

   (b) replace x by $x + t$, where t is any real number.

   By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

5. Let n be a positive integer. In a group of $2n + 1$ people, each pair is classied as friends or strangers. For every set S of at most n people, there is one person outside of S who is friends with everyone in S. Prove that at least one person is friends with everyone else.

6. There are n points in the plane such that no three of them are collinear. Prove that the number of triangles, whose vertices are chosen from these n points and whose area is 1, is not greater than $\frac{2(n^2-n)}{3}$.

7. Show that

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}.$$

8. Let $f_k(n)$ be the number of permutations of $\{1, \cdots, n\}$ that have exactly k fixed points. Show

$$\sum_{k=0}^{n} kf_k(n) = n!.$$