Winter Quarter Week 2: Combinatorics

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1. Prove that among any six people, there are three who are all friends with each other or three who are all not friends with each other. Is the same true for five people?

2. (Italy 2007) Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are to colour the computers and cables such that no two computers have the same colour, no two cables joined to a common computer have the same colour, and no computer is assigned the same colour as any cable joined to it. Prove that this can be done using n colours.

3. (BAMO 2014) A chess tournament took place between $2n+1$ players. Every player played every other player once, with no draws. In addition, each player had a numerical rating before the tournament began, with no two players having equal ratings.

   It turns out there were exactly k games in which the lower-rated player beat the higher-rated player. Prove that there is some player who won no less than $n - \sqrt{2k}$ and no more than $n + \sqrt{2k}$ games.

4. Some number of frogs are squatting on a row of 2000 lily pads in a swamp. Each minute, if there are two frogs on the same lily pad, and this pad is not at either end of the row, the two frogs may jump to two adjacent lily pads (in opposite directions). Prove that this process cannot be repeated forever.

5. Ivan has a 52-card deck. He draws the cards from the deck one by one, without putting them back in the deck. Every time before drawing a card he guesses the suit of the card he will draw. He decides to always guess the suit that occurs most frequently in the remaining deck (if there are several such suits, he chooses any one of them). Prove that he will guess the right suit at least 13 times.

6. Are those true or false? Provide proof or counterexample.

   a) In a 5x5 chessboard a square is removed. Is it possible to tile the resulting board with dominoes?

   b) Two opposite corners are removed from a standard chessboard. Is it possible to tile the resulting board with dominoes?

   c) A knight is placed on the vertex of an nxn board. What are the possible squares it can go to in less than $8n^2$ moves.

7. The nonzero numbers $a_1, \cdots, a_n \in \mathbb{R}$ are written on the board. One can make a "move" by replacing two numbers a, b by $\frac{a+b}{2}$ and $\frac{a-b}{2}$. Show that after any finite number of moves one can never come back to the same list of number that we started with.

8. The numbers $1, \cdots, n$ are written on the board. We are allowed to swap any two adjacent numbers.

   a) Show that every permutation can be achieved in finitely many steps.

   b) If 2007 swaps are performed, is it possible that the final arrangement of numbers coincides with the original?

9. Let n be odd and $a_1, \cdots, a_n$ permutation of $1, \cdots, n$. Show that $(a_1 - 1) \cdots (a_n - n)$ is even.

Remark: In an invariant problem, some of the most useful things to look at are: possible colorings, algebraic expressions, and inversions.