Olympiad Group Winter Break: Problems from Number Theory and Intro to Combinatorics

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1 Number Theory Problems

1. Find all positive integers n so that $5^n | (101^n - 1)$.
2. Assume that $a > 1$ is even. Show that $\{a^{a^n} + 1\}_{n \in \mathbb{N}}$ are pairwise relatively prime.
3. Show that if a and b are relatively prime positive integers, then there exists integers m and n such that $a^m + b^n \equiv 1 \pmod{ab}$.
4. (Harder) Show there are infinitely many primes of the form $6k + 1$.

2 Choosing Objects

Perhaps the simplest and most natural counting problems goes as follows: given n distinct objects, how many ways are there to choose some subset of $0 \leq k \leq n$ of them? This problem applies to real world situations like drawing a hand of cards, or choosing toppings to put on a pizza.

In order to better understand this problem, we will write

$$n \text{ choose } k = \binom{n}{k}$$

to denote the number of ways to choose k objects from n distinct objects; formally, it is the number of subsets of size k of a set of size n. You should already know the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The goal of this section is to see how far we can get without the formula. Most of the problems in this section are easy with the factorial formula, but the point is to build intuition by working with the original definition, and to gain appreciation for the power of the formula. Do not use the formula in any of your solutions!

2.1 Problems

1. (Don't turn this in) Write out the first few values of $\binom{n}{k}$ in a triangle, with n increasing top to bottom and k increasing left to right. This is called Pascal's Triangle. Can you see any relationships that the values in the triangle satisfy?
2. Prove that the sum of the entries in the row corresponding to n in Pascal's triangle is 2^n.

3. Show that $\binom{n}{k}$ counts the number of ways to move from the point $(0,0)$ to the point $(k, n-k)$ on the lattice of points with integer coordinates, where each move is either to go up by 1 or right by 1.

4. Prove Pascal's Identity:
$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

5. I (Jacob) took my phone and timed myself writing the first 10 rows of Pascal's triangle (up to the $\binom{9}{n}$ row) in 45.3 seconds. Do this and record how long it takes you.

6. In a 5 by 5 square lattice with the central point removed, how many ways are there to travel from the bottom left to the top right, moving only right by 1 or up by 1 each move?

7. Prove the Binomial Theorem by induction. In case you forgot, the statement is:
$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + y^n$$

8. If p is a prime number and $1 \leq k \leq p-1$, show that p divides $\binom{p}{k}$. Find a way to argue this without the factorial formula!

9. Prove Fermat's little theorem by induction. In case you forgot, the statement is: if p is a prime number and x is any integer, then p divides $x^p - x$.

3 Permutations

Given n distinct objects, there are $n! = n(n-1)\cdots(2)1$ ways to put them in order with one object first, another object second, up to the last object nth. An ordering like this is called a permutation.

Introducing permutations and the factorial function allow us to write the up until now mysterious values $\binom{n}{k}$ explicitly. Consider a subset S of k out of n objects. The number of permutations of the n objects such that the first k objects are exactly those in S is $k!(n-k)!$. The total number of permutations of the n objects is $n!$. Therefore, there are
$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$
total different subsets k of the n objects.

3.1 Problems

1. Show that there are $\frac{n!}{(n-k)!}$ ways to pick k objects in order from a set of n. This number is sometimes called "n pick k".

2. How many ways are there to rearrange the letters of $LAMATHCIRCLE$? Leave your answer in factorial notation.