# Point Mass Geometry I Solutions

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1. We put a mass of 1 on B and C, so that 2D is the midpoint of the segment BC. Then put a mass of 2 on A so that Z is the midpoint of 2D and 2A. Therefore, Z is the center of mass of the triangle (1B)(1C)(2A). Now take the center of mass of 2A and 1C, giving you a point 3F. But the overall center of mass of the system Z is now on the line BF, so F is the point we want. Therefore, F divides the side AC into a ratio of 1:2.

2. Give the system masses 2A, 1C, 1N. Then it follows that 3M is the center of mass of 2A and 1C. Therefore, the center of mass of the system lies on the line MN. However, if we first take the center of mass of 1N and 1C, we get 2B, so the center of mass lies on the line AB. Therefore, the center of mass of the system must be P. Furthermore, P is the midpoint of 2A and 2B, so $|AP| : |PB| = 1 : 1$, and we have P is the center of mass of 1N and 3M, so $|NP| : |PM| = 3 : 1$.

3. Label the system 1A, 1B, 1C, 1D. Then we have 2K, 2L, 2M, 2N are the midpoint of their respective sides. Now by taking first the center of mass of AB and CD, we get 2K and 2M. Then taking the center of mass of those we get that the center of mass of the system O is the midpoint of KM. Similarly, by doing BC and DA first, we get that the center of mass of the system O is the midpoint of LN. But the center of mass is unique, so it must be the intersection of KM and LN. Alternatively, taking first the center of mass of AC and BD, we get the midpoints of the diagonals. Then taking the center of mass of those two points, we have that O is the midpoint of the segment connecting the midpoints of the diagonals (since they both had mass 2).

4. Label the system 1B, 1C, 3A. Then we have 2D and 4E. Then $K = BE \cap AD$ is the center of mass of the system since combining A and C first, the center of mass lies on BE and if you combine B and C first, it lies on AD. Then we have that $|AK| : |KD| = 2 : 3$ and $|BK| : |KE| = 4 : 1$.

5. We take the hint. We label the system 1A, 2B, 1C, so that the center of mass is the midpoint L of 2B and $2B_1$. Then if we instead first take the center of mass of 2B and 1C, we have that the extension of AL divides BC into a ratio of 1:2. Now consider a new system 2A, 2B, 1C. Then the center of mass of this system still lies on AL, but it also lies on $CC_1$, since taking first the center of mass of 2A and 2B gives $4C_1$. Then we have that AL divides $CC_1$ into a ratio of 4:1.

6. We use the fact that the diagonals of a parallelogram bisect each other and ignore the point C for the time being. We set up the triangle $\triangle ABD$ so that K is the center of mass of A and B. So assign masses (n-1)A, 1B, 1D, so that the center of mass of the triangle is on both DK and AO, where O is the intersection of the diagonals of the parallelogram (the midpoint of $BD$). Let us call the center of mass P, this is where the line DK intersects the diagonal AC. Then we have (n-1)A and 2O, so $|AP| : |PO| = 2 : n - 1$. But $|AO| = |OC|$, so $|AP| : |PC| = 2 : 2n = 1 : n$.

7. Assign the vertices of the quadrilateral ABCD masses of 1 each. Then we have that the midpoints are given by 2E, 2F, 2G, 2H, where E is the midpoint of AB, and so on. Then the center of mass of this new quadrilateral is the midpoint of EG and also the midpoint of FH. Therefore, its diagonals bisect each other and so EFGH is a parallelogram.

8. Notice that $|AM| = |AQ|, |BN| = |BM|, |CP| = |CN|, |DQ| = |DP|$. We start assigning masses to the vertices A, B, C, and D. Let A have mass 1, then to make M the center of mass of A and B, we give B a mass of $\frac{a}{b}$. This forces C to have a mass of $\frac{a}{c}$ and D to have a mass of $\frac{a}{d}$. Then you can show that Z is the center of mass of the system, sitting on both MP and NQ. Then it follows that $|MZ| : |ZP| = \frac{a}{d} + \frac{a}{c} : 1 + \frac{a}{b}$, and $|QZ| : |ZN| = \frac{a}{b} + \frac{a}{c} : 1 + \frac{a}{d}$.

9. Label the system 1A, 3C, 6B. Then it follows that 4M is the center of mass of A and C and 9P is the center of mass of B and C. Then since $3|BP| = |BC|$, and $10|PQ| = |PA|$, you can show that the height of $\triangle ABC$ is 10 times larger than the height of $\triangle BPQ$. Taking $BC$ to be the base, we see that the area of $\triangle ABC = 30Area(\triangle BPQ) = 30$.

10. Let $A_1, B_1, C_1$ be the intersection points of the plane $\alpha$ with the sides PA, PB, and PC, respectively. Then if we put point masses 12A, 12B, 12C, 18P, 8P, 3P, then taking the appropriate pairwise center of masses, we get $30A_1, 20B_1, 15C_1$. The center of mass of the system is then the center of mass of $30A_1, 20B_1, 15C_1$, which lies on the plane $\alpha$. However, if we first combine the point masses at P to $29P$ and the point masses at A,B, and C to $36M$, where M is the center of the base triangle, then the center of mass must also lie on the altitude PM. Therefore, the plane divides the altitude into the ratio 36 : 29.