Point Mass Geometry II

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Warm-up problems

Use point mass geometry to solve the following problems.

1. The base $ABCD$ of a pyramid $FABCD$ is a parallelogram. The plane α intersects AF, BF, CF and DF at points A_1, B_1, C_1, D_1 respectively. Given that

$$\frac{|AA_1|}{|A_1F|} = 2, \quad \frac{|BB_1|}{|B_1F|} = 5, \quad \frac{|CC_1|}{|C_1F|} = 10,$$

find the ratio $r = \frac{|DD_1|}{|D_1F|}$.
Let $O = AC \cap BD$. Let K be the point of intersection of FO and the plane α. Use the following plan to solve the problem.

   (a) Notice that K is the point of intersection of FO and A_1C_1.
   - Place point masses at A and F in such a way that A_1 is their center of mass.
   - Place point masses at F and C in such a way that C_1 is their center of mass.
   - Point K is the center of mass of the system of these 4 masses (two of which are placed at F). Use this to find the ratio $|FK| : |KO|$.

   (b) Notice that K is also the point of intersection of FO and B_1D_1. Let r be the ratio $r = \frac{|DD_1|}{|D_1F|}$.
   - Place point masses at points B and F in such a way that $B_1$is their center of mass.
   - Place point masses at F and D in such a way that D_1 is their center of mass.
   - Point K is the center of mass of the system of these 4 masses (two of which are placed at F). Use the ratio $|FK| : |KO|$ that you found in part (a) of the problem to determine r.

2. Let $L \in AC$ and $M \in BC$ be the points on the sides of $\triangle ABC$ such that

$$|CL| = \alpha \cdot |CA|, \quad |CM| = \beta \cdot |CB|, \qquad \text{where } \ 0 < \alpha, \beta < 1).$$

Let $P = AM \cap BL$. Find the ratio $\frac{|AP|}{|AM|}$.

Center of mass via vectors

- Let $Z = Z(m_1A,\ m_2B)$ be the center of mass of point masses m_1 and m_2 placed at points A and B. Then the law of levers $m_1d_1 = m_2d_2$ can be written as
$$m_1|\overrightarrow{ZA_1}| \;=\; m_2|\overrightarrow{ZA_2}|.$$
- Since the vectors $\overrightarrow{ZA_1}$ and $\overrightarrow{ZA_2}$ have opposite directions, it follows that
$$m_1\overrightarrow{ZA_1} + m_2\overrightarrow{ZA_2} = 0.$$

  This condition can be taken as the definition of the center of mass of the system of two points. In the case when there are more then two masses, we get the following

Definition.

The center of mass of the system of n point masses $m_1A_1, m_2A_2, \ldots m_nA_n$ is the point Z such that
$$m_1\overrightarrow{ZA_1} + m_2\overrightarrow{ZA_2} + \ldots + m_n\overrightarrow{ZA_n} = 0.$$

1. Show that if $Z = Z(m_1A_1, m_2A_2)$, then for any point O we have
$$\overrightarrow{OZ} = \frac{m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2}}{m_1 + m_2}.$$
2. Suppose that for any point O on the plane and a point Z the equality above holds. Is it true that Z is the center of mass of m_1A_1 and m_2A_2?
3. Let G be the point of intersection of medians of $\triangle ABC$. Find $\overrightarrow{AG}$ in terms of $\overrightarrow{AC}$ and $\overrightarrow{AB}$.
4. Let C be the point on a segment AB such that $|AC| : |CB| = 2 : 7$. Express this fact using the notion of the center of mass. Express the same fact using vectors.
5. Let G and G_1 be the centers of mass of $\triangle ABC$ and $\triangle A_1B_1C_1$ respectively. Show that
$$\overrightarrow{GG_1} = \frac{1}{3}(\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1}).$$

Negative masses

The same definition of the center of mass works if some (or all) of the masses in the system are negative.

Definition.

Let m_1A_1 and m_2A_2 be two point masses. Assuming that $m_1 + m_2 \neq 0$, the center of mass of this system $Z = Z(m_1A_1, m_2A_2)$ lies on the line A_1A_2 so that

$$|m_1| \cdot d_1 \; = \; |m_2| \cdot d_2,$$

where $d_1 = |\overrightarrow{ZA_1}|$ and $d_2 = |\overrightarrow{ZA_2}|$.

The center of mass Z lies between A_1 and A_2 if and only if the masses have the same sign (i.e., both are positive or both are negative).

1. Show that if $ABCD$ is a parallelogram, then $mD = Z(mA,\ (-m)B,\ mC)$.

2. The base of a pyramid $SABCD$ is a parallelogram $ABCD$. A plane α intersects the sides SA, SB, SC and SD at points A_1, B_1, C_1, D_1 respectively. Given that $|SA_1| = \frac{1}{3}|SA|$, $|SB_1| = \frac{1}{5}|SB|$ and $|SC_1| = \frac{1}{4}|SC|$, find the ratio $|SD_1|/|SD|$ (use the previous problem).

3. Solve problem 1 from page 1 using masses of different signs. Which solution do you prefer?

Barycentric coordinates

- Let M be a point inside of $\triangle ABC$. One can find masses m_1, m_2, m_3 so that $M = Z(m_1A,\ m_2B,\ m_3C)$;

- The masses m_1, m_2, m_3 are defined up to a constant factor $k \neq 0$: $Z(km_1A,\ km_2B,\ km_3C) = Z(m_1A,\ m_2B,\ m_3C)$;

- Define k so that the sum of masses equals to 1.

Definition of barycentric coordinates

For any point M inside of $\triangle ABC$ there are positive numbers μ_1, μ_2, μ_3 so that

- $\mu_1 + \mu_2 + \mu_3 = 1$;

- $M = Z(\mu_1A,\ \mu_2B,\ \mu_3C)$.

*These numbers are called the **barycentric coordinates** (or, **B-coordinates) of** M with respect to $\triangle ABC$.*

Warm-up problems on barycentric coordinates

1. Find the barycentric coordinates of each of the vertices.

2. Find the point with barycentric coordinates with respect to $\triangle ABC$ are equal to $(1/2,\ 1/2,\ 0)$.

3. Find the barycentric coordinates of the point of intersection of medians.

4. Let μ_1, μ_2, μ_3 be the barycentric coordinate of a point M with respect to $\triangle ABC$. Prove the following statements:

   (a) M lies on the line through points A and B if and only if $\mu_3 = 0$;

   (b) M lies on the segment AB if and only if $\mu_1, \mu_2 > 0$ and $\mu_3 = 0$.

5. Draw a triangle $\triangle ABC$ and mark the following points with given barycentric coordinates: $M = (1/2,\ 1/4,\ 1/4)$, $N = (-1,\ 2,\ 0)$.

6. Let $M \in BC$ be such that $|BM| = \frac{1}{3}|BC|$. Let $N \in AB$ be such that $|AN| = \frac{1}{3}|AB|$. Find the barycentric coordinates of the point of intersection of AM and CN.

More problems

1. Show that the numbers μ_1, μ_2, μ_3 satisfying conditions in the definition of barycentric coordinates exist and are unique for any point M on the plane. (*Hint:* Use the fact that M is the center of mass of A, B, C if and only if for any point P we have

$$\overrightarrow{OM} = \mu_1 \overrightarrow{OA} + \mu_2 \overrightarrow{OB} + \mu_3 \overrightarrow{OC}.$$

   Choose point O in a smart way to prove the statement).

2. Based on the previous problem, describe a (graphical) way to determine the barycentric coordinates of any point M with respect to a given triangle $\triangle ABC$.

3. Find the barycentric coordinates of the point of intersection of altitudes of acute triangle $\triangle ABC$ given that its sides have lengths a, b, c and the angles are equal to α, β, γ.

4. Let (m_1, m_2, m_3) and (n_1, n_2, n_3) be the barycentric coordinates of points M and N respectively. Find the barycentric coordinates of the mid point of the segment MN.

5. (Barycentric coordinates as areas): Let P be a point inside of $\triangle A_1A_2A_3$. Let $S,\ S_1,\ S_2,\ S_3$ be the areas of triangles $\triangle A_1A_2A_3$, $\triangle PA_2A_3$, $\triangle PA_1A_3$ and $\triangle PA_1A_2$ respectively. Then the barycentric coordinates of P are:

$$\mu_1 = \frac{S_1}{S}, \quad \mu_2 = \frac{S_2}{S}, \quad \mu = \frac{S_3}{S}.$$

(In other words, if $\triangle ABC$ has unit area, P is the center of mass of three masses positioned at the vertices where the masses are chosen proportionately to indicated areas).

6. Find the barycentric coordinates of the center of inscribed circle of $\triangle ABC$. (*Hint:* use interpretation of barycentric coordinates in terms of areas; see the previous problem).

7. Let B_1, B_2, B_3 be points on the sides of triangle $\triangle A_1A_2A_3$ such that

$$\frac{|A_2B_1|}{|A_2A_3|} = \frac{|A_3B_2|}{|A_3A_1|} = \frac{|A_1B_3|}{|A_1A_2|} = \frac{1}{4}.$$

   Let S be the area of triangle ABC. Find the area of the triangle PQR bounded by the lines A_1B_1, A_2B_2, A_3B_3.

The problems are taken from:

M. Bulk, V. Boltyanskij, "Geometry of Mass", 1987 (in Russian)