Lesson 8: Games and Geometry III

Konstantin Miagkov

June 26, 2019

Remark 1.
The problems in this lesson concern the technique of "stealing strategies", where the first player cannot lose if they can skip a turn and essentially choose whether they want to be the first of second player.

Problem 1.
Show that in a game of tic-tac-toe on an infinite board the second player does not have a winning strategy. In infinite tic-tac-toe one needs 5 in a row to win.

Proof. Suppose the second player has a winning strategy. Let us now describe a winning strategy for the first player, which would be a contradiction. Let the first player make an arbitrary first move, and then pretend they are the second player and follow the second player's winning strategy. There is an extra X on the board – the first move of the first player, but the first player should just forget that it is there and ignore it. The only time that is not possible is when the first player would like to place an X on that exact square according to his strategy. But then that X is already placed, so we simply place an X on a random empty square. Therefore the position on the board is always the position according to the winning strategy of the second player, except the first player is second and there is one extra X on the board. Since the strategy is winning, eventually there will be five X's in a row, which would still be true with an extra X (if anything, it might happen sooner). □

Problem 2.
Two players are playing a game at night on the streets of the Candy Kingdom. The streets of the Candy Kingdom make a rectangular grid. Every turn consists of finding a not yet lit intersection, and putting a projector there, which lights up everything to the top and right of itself (including the intersection it is on). The person, after whose move the whole kingdom is lit for the first time loses. Who has a winning strategy?

Proof. Since the game is finite, somebody has a winning strategy. Suppose it is the second player. Note that their strategy cannot involve ever putting a light to the top-right corner of the board, since that intersection is always already lit after the first move of the first player. Now let us describe the winning strategy of the first player – they first put a light in the top-right corner, and then follow the winning strategy of the first player. This is possible since that winning strategy never uses the already-occupied top-right corner. Thus the first player, wins, contradiction, which means they must have a winning strategy in the first place. □

Problem 3.
Kiselev 251, page 96.

Proof. The locus is easily seen to be a circle centered at the center of the given circle, by computing the distance from the point on the locus to that center as the sum of the given radius and the radius of the given circle. □

Problem 4.
Kiselev 253, page 96.

Proof. Suppose the shortest distance is achieved by two points A and B such that AB is not the line between the centers. Let A' and B' be the points on the corresponding circles on ht line between the centers, and let O and Q be the centers. Then $OA = OA'$ and $QB = QB'$, and by the generalized triangle inequality (or just the triangle inequality applied twice) we have

$$OA + AB + BQ < OQ = OA' + A'B' + B'Q$$

$$AB < A'B'$$

which is a contradiction. □

Problem 5.
Suppose n points are marked on the plane, where $n \geq 9$. It is known that for any 9 of the points one can draw two circles so that all 9 points lie on those circles. Show that it is possible two draw two circles so that all n points lie on them.

Proof. Pick any 9 points. Then they lie on two circles, and thus one of the circles contains at least 5 points. Call that circle ω. Take any 9 points which contain the 5 points on ω. Those 9 points must lie on two circle, and so one of the two must contain at least 3 points out of the 5 on omega. But then that circle must coincide with ω. Thus for any 9 points containing the 5 on ω, one of the two circles they lie on must be ω. Thus if we take any 4 points not on ω, those together with the 5 points on ω must lie on two circles, and one of them is ω, implying that the 4 points lie on a circle. Thus if we forget about all points on ω, we have the following problem – any 4 points lie on a circle, show that all points lie one a single circle. But this is easy – pick any three points, and any 4th points must lie on the circumcircle of these 3 – thus all points lie on that circumcircle. □