Homework 6: Games and Geometry

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Problem 1.
Consider the system of equations:

$$\begin{cases} *x + *y = 0 \\ *x + *y = 0 \end{cases}$$

Two players take turns replacing the stars with real numbers. The goal of the first player is to make the system have no nontrivial solutions after all the stars have been replaced, and the goal of the second player is to make the system have at least one nontrivial solution. Who has a winning strategy? (A nontrivial solution means a solution which is not (0, 0))

Proof. The second player wins. If the first player sets a coefficient by x or y in any of the equations to a, we will set the coefficient by x or y (whichever is left) in the same equation to a as well. It is easy to see that after the game is over, $(1, -1)$ will be a solution, so the second player wins. □

Problem 2.
On a chord AR, two points are taken the same distance away from the midpoint C of this chord, and through these points, two perpendiculars to AR are drawn up to their intersections with the circle. Prove that these perpendiculars are congruent. Hint: Fold the diagram along the diameter passing through C.

Proof. Let us label the points chosen on the cord as M and N, where M is on the segment CA. Then let the intersections of the perpendicular through M with the circle be P_1 and P_2, and the intersections of the perpendicular through N with the circle be Q_1 and Q_2. Let O be the center of the circle, and draw a line through O parallel through AB. Let the intersections of that line with P_1P_2 and Q_1Q_2 be K and T respectively. Then $KMCO$ and $TNCO$ are rectangles, so $KO = MC = NC = TO$. Then $\triangle OKP_1$ and $\triangle OTQ_1$ are right triangles with $OT = OK$ and $OP_1 = OQ_1$, thus they are congruent and $KP_1 = TQ_1$. Similarly $\triangle OKP_2 = \triangle OTQ_2$ and so $KP_2 = TQ_2$. Therefore we get $P_1P_2 = Q_1Q_2$, and we are done. □