Point Mass Geometry (part II)

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In each of the problems below the key is to put some masses at some of the points in the problem

More problems:

1. The base $ABCD$ of a pyramid $FABCD$ is a parallelogram. The plane α intersects AF, BF, CF and DF at points A_1, B_1, C_1, D_1 respectively. Given that
$$\frac{|AA_1|}{|A_1F|} = 2, \quad \frac{|BB_1|}{|B_1F|} = 5, \quad \frac{|CC_1|}{|C_1F|} = 10,$$
find the ratio $x = \frac{|DD_1|}{|D_1F}$.

2. Let $L \in AC$ and $M \in BC$ be the points on the sides of $\triangle ABC$ so that
$$|CL| = \alpha|CA|, \quad |CM| = \beta|CB| \qquad (0 < \alpha, \beta < 1).$$
Let $P = AM \cap BL$. Find the ratio $\frac{|AP|}{|AM}$.

Center of Mass via vectors:

- Let $Z = Z(m_1A,\ m_2B)$. Then the condition $m_1d_1 = m_2d_2$ can be written as
$$m_1|\overrightarrow{ZA_1}| \ = \ m_2|\overrightarrow{ZA_2}|.$$
Since $\overrightarrow{ZA_1}$ and $\overrightarrow{ZA_2}$ have opposite directions, it follows that
$$m_1\overrightarrow{ZA_1} + m_2\overrightarrow{ZA_2} = 0.$$

- This condition can be taken as the definition of the center of mass of the system of two points.

In the case when there are more then two masses, we get the following:

The center of mass of the system of n point masses $m_1A_1, m_2A_2, \ldots m_nA_n$ is the point Z such that
$$m_1\overrightarrow{ZA_1} + m_2\overrightarrow{ZA_2} + \ldots + m_n\overrightarrow{ZA_n} = 0.$$

1. Show that if $Z = Z(m_1A_1, m_2A_2)$, then for any point O we have

$$\overrightarrow{OZ} = \frac{m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2}}{m_1 + m_2}.$$

   Is it true that if for any point O and a point Z the equality above holds, then Z is the center of mass of A_1 and A_2?

2. Let G be the point of intersection of medians of $\triangle ABC$. Find $\overrightarrow{AG}$ in terms of $\overrightarrow{AC}$ and $\overrightarrow{AB}$.

3. Let C be the point on a segment AB such that $|AC| : |CB| = 2 : 7$. Expess this fact using the notion of the center of mass. Express the same fact using vectors.

4. Let G and G' be the centers of mass of $\triangle ABC$ and $\triangle A_1B_1C_1'$ respectively. Show that

$$\overrightarrow{GG_1} = \frac{1}{3}(\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1}).$$

Negative masses:

The same definition of the center of mass works if some (or all) of the masses in the system are negative.

- Let m_1A_1 and m_2A_2 be two point masses. Assuming that $m_1 + m_2 \neq 0$, the center of mass of this system $Z = Z(m_1A_1, m_2A_2)$ lies on the line A_1A_2 so that

$$|m_1| \cdot d_1 \;=\; |m_2| \cdot d_2,$$

  where $d_1 = |ZA_1|$ and $d_2 = |ZA_2|$. The center of mass Z lies between A_1 and A_2 if and only if the masses have the same sign (i.e., both are positive or both are negative).

1. Show that if $ABCD$ is a parallelogram, then $mD = Z(mA,\ (-m)B,\ mC)$.

2. The base of a pyramid $SABCD$ is a parallelogram $ABCD$. A plane α intersects the sides SA, SB, SC and SD at points A_1, B_1, C_1, D_1 respectively. Given that $|SA_1| = \frac{1}{3}|SA|$, $|SB_1| = \frac{1}{5}|SB|$ and $|SC_1| = \frac{1}{4}|SC|$, find the ratio $|SD_1|/|SD|$ (use the previous problem).

Barycentric coordinates:

- Let M be point inside of $\triangle ABC$. One can find masses m_1, m_2, m_3 so that $M = Z(m_1A,\ m_2B,\ m_3C)$;

- The masses m_1, m_2, m_3 are defined up to a constant factor: $Z(km_1A, km_2B, km_3C) = Z(m_1A, m_2B, m_3C)$;

- Define k so that the sum of masses equals to 1.

For any point M inside of $\triangle ABC$ there are positive numbers μ_1, μ_2, μ_3 so that

- $\mu_1 + \mu_2 + \mu_3 = 1$;
- $M = Z(\mu_1 A,\ \mu_2 B,\ \mu_3 C)$.

These numbers are called the ***baricentric coordinates*** *(or,* ***B-coordinates) of*** *M with respect to $\triangle ABC$.*

Warm-up problems:

1. Find the baricentric coordinates of each of the vertices.
2. Find the point whose baricentric coordinates with respect to $\triangle ABC$ are equal to $(1/2,\ 1/2,\ 0)$.
3. Find the baricentric coordinates of the point of intersection of medians.
4. Let μ_1, μ_2, μ_3 be the baricentric coordinate of a point M with respect to $\triangle ABC$. Prove the following statements:
   (a) M lies on the line trough A, B if and only if $\mu_3 = 0$;
   (b) M lies on the segment $[AB]$ if and only if $\mu_1, \mu_2 > 0$ and $\mu_3 = 0$.
5. Draw a triangle $\triangle ABC$ and mark the following points with given baricentric coordinates: $M(1/2,\ 1/4,\ 1/4)$, $M(-1, 2, 0)$.
6. Let $M \in BC$ be such that $|BM| = 1/3|BC|$. Let $N \in AB$ be such that $|AN| = \frac{1}{3}|AB|$. Find the baricentric coordinates of the point of intersection of AM and CN.

Problems:

1. Show that the numbers μ_1, μ_2, μ_3 satisfying conditions in the definition of baricentric coordinates exist and are unique for any point M on the plane. (*Hint:* Use the fact that M is the center of mass of A, B, C if and only if for any point P we have

$$\overrightarrow{OM} = \mu_1\overrightarrow{OA} + \mu_2\overrightarrow{OB} + \mu_3\overrightarrow{OC}.$$

   Choose point O in a smart way to prove the statement).
2. Based on the previous problem, describe a (graphical) way to determine the baricentric coordinates of any point M with respect to a given triangle $\triangle ABC$.
3. Find the baricentric coordinates of the point of intersection of altitudes of (acute angle) triangle $\triangle ABC$ given that it's sides have lengths a, b, c and the angles are equal to α, β, γ.

4. Let (m_1, m_2, m_3) and (n_1, n_2, n_3) be the baricentric coordinates of M and N respectively. Find the baricentric coordinates of the mid point of the segment MN.

5. (Baricentric coordinates as areas): Let P be a point inside of $\triangle A_1A_2A_3$. Let S, S_1, S_2, S_3 be the areas of triangles $\triangle A_1A_2A_3$, $\triangle PA_2A_3$, $\triangle PA_1A_3$ and $\triangle PA_1A_2$ respectively. Then the baricentric coordinates of P are:

$$\mu_1 = \frac{S_1}{S}, \quad \mu_2 = \frac{S_2}{S}, \quad \mu = \frac{S_3}{S}.$$

   (In other words, if $\triangle ABC$ has unit area, P is the center of mass of three masses positioned at the vertices where the masses are chosen proportinately to indicated areas).

6. Find the baricentric coordinates of the center of the inscribed circle of $\triangle ABC$. (*Hint:* use interpretation of baricentric coordinates in terms of areas; see the previous problem).

7. Let B_1, B_2, B_3 be points on the sides of triangle $\triangle A_1A_2A_3$ such that

$$\frac{|A_2B_1|}{|A_2A_3|} = \frac{|A_3B_2|}{|A_3A_1|} = \frac{|A_1B_3|}{|A_1A_2|} = \frac{1}{4}.$$

   Let S be the area of triangle ABC. Find the area of the triangle PQR bounded by the lines A_1B_1, A_2B_2, A_3B_3.

The problems are taken from:

M. Bulk, V. Boltyanskij, "Geometry of Mass", 1987 (in Russian)