Lesson 7: Games and Geometry II

Konstantin Miagkov

Remark 1.
In the beginning of this class it is necessary to introduce winning and losing positions (also called N and P positions). It is beneficial to show an example of computing such positions – for example, using problem L6.1.

Problem 1.
a) Two players are taking turns moving a *limp king* on a 7×7 board. A limp king is a chess piece which can move one square to the right or diagonally to the right and up. The limp king starts at the bottom-left corner of the board. The player who cannot make a turn loses. Who has a winning strategy?

Proof. The first player always wins, the moves do not matter. □

b) Same problem, but now the limp king can also go up one square.

Proof. If we analyze winning and losing positions, we get that a position (x, y) on the board is losing if both x and y are odd. Thus $(1, 1)$ is losing and the second player wins. □

Problem 2.
There is a bunch of 10 million matches. Two players are playing the next game. They take turns. In one turn, the player can take p^n matches from the bunch, where p is a prime number and $n = 0, 1, 2, 3, \dots$ (for example, the first takes 25 matches, the second 8, the first 1, the second one 5, the first - 49, etc.). The one who takes the last match wins. Who will win if both players play optimally?

Proof. The first player wins. All losing positions are exactly positions where the number of the matches is divisible by 6. This can be proved inductively. Indeed, from each winning position one can move to a losing one since 1, 2, 3, 4, and 5 are the powers of primes. However, one can not move from one losing position to another since no power of a prime is divisible by 6. □

Problem 3.
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Problem 4.
Find the geometric locus of points from which the tangents drawn to a given circle are congruent to a given segment.

Proof. Let A, B be two points such that the tangents from A and B to a given circle are equal. Let O be the center of the circle, and let A', B' be the tangency points of tangents from A, B to the circle. Then $\triangle AA'O$ and $\triangle BB'O$ are both right triangles with $AA' = BB'$ and $A'O = B'O$, thus they are congruent. Therefore $AO = BO$. Thus the answer is a circle centered at O. We also have to show the other direction – that $AO = BO$ implies $AA' = BB'$, but that also follows from the same triangle congruence. □

Problem 5.
a) Two lines passing through a point M are tangent to a circle at the points A and B. Through a point C taken on the smaller of the arcs AB, a third tangent is drawn up to its intersection points D and E with MA and MB respectively. Prove that the perimeter of $\triangle\ DME$ does not depend on the position of the point C. Hint: The perimeter is congruent to $MA + MB$;

b) Prove that the angle $\angle DOE$ (where O is the center of the of circle) does not depend on the position of the point C.

Hint: $\angle DOE = \frac{\angle AOB}{2}$

Proof. We have $DC = DA$ and $EC = EB$. Then perimeter of $\triangle MDE$ is

$$MD + ME + DE = MD + ME + DA + EB = MA + MB$$

We also have $\triangle OCD = \triangle OAD$ and thus $\angle COD = \angle AOD$ and similarly $\angle COE = \angle BOE$. Then

$$\angle DOE = \angle COD + \angle COE = \frac{1}{2}\angle AOB$$

□