COMPLEX NUMBERS

LAMC OLYMPIAD GROUP, WEEK 8

Complex numbers are of the form $a + bi$, where a and b are real numbers, and i is a formal symbol with the property that $i^2 = -1$ (note that no real number satisfies this property):

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

This isn't very rigorous: formally we could consider pairs of real numbers (a, b) and define a special type of addition and multiplication on such pairs, identifying $(1, 0)$ with 1 and $(0, 1)$ with i (but no one thinks of complex numbers that way). When you add and multiply complex numbers, just apply the usual rules together with the novel $i^2 = -1$:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

We have $a+bi = c+di$ if and only if $a = c$ and $b = d$ (where $a, b, c, d \in \mathbb{R}$); this makes sense because $(a-c) = (d-b)i$ implies $(a - c)^2 = -(d - b)^2$, which forces both squares to be zero (notice the similarity with the same proof for $a + b\sqrt{2} = c + d\sqrt{2}$ with $a, b, c, d \in \mathbb{Z}$). For $a, b \in \mathbb{R}$, we call a the *real part* of $a + bi$ and b the *imaginary part* of $a + bi$:

$$\operatorname{Re}(a + bi) = a, \qquad \operatorname{Im}(a + bi) = b.$$

For $a + bi \neq 0$ (that is, not both a and b are 0), we can also invert it:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

Above, we used that $(a+bi)(a-bi) = a^2+b^2$. Here, $a-bi$ is called the *conjugate* of $a+bi$ (denoted with an overline), and $\sqrt{a^2 + b^2}$ is called the *absolute value* (or modulus) of $a + bi$:

$$\overline{a + bi} = a - bi, \qquad |a + bi| = \sqrt{a^2 + b^2}.$$

So for any complex number z, we have

$$z \cdot \overline{z} = |z|^2, \qquad \operatorname{Re} z = \frac{z + \overline{z}}{2}, \qquad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

In particular, z is *real* if and only if $z = \overline{z}$, and z is *imaginary* (real part = 0) if and only if $z = -\overline{z}$.

Complex numbers can be viewed in a plane (the *complex plane*), where $z = a + bi$ corresponds to the point of coordinates (a, b). Then the absolute value $|z|$ represents the length of the segment connecting the origin 0 to the point z (from this picture, it is easy to notice the triangle inequality $|z+w| \leq |z|+|w|$). If $z \neq 0$, the *angle* in counterclockwise direction formed by this segment with the real x-axis is denoted by $\arg z$ (think of this as a number mod 2π). Then

$$z = r(\cos\varphi + i\sin\varphi),$$

where $r = |z|$ and $\varphi = \arg z$. In this writing (called the polar representation), it is very easy to multiply complex numbers; absolute values are multiplied, and arguments are added:

$$|zw| = |z| \cdot |w|, \qquad \arg(z \cdot w) = \arg z + \arg w,$$

where adding arguments is done mod 2π (so 3π is the same as π, for example). This follows from the fact that $|\cos\varphi + i\sin\varphi| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$ and from the formulae for $\cos(x + y)$ and $\sin(x + y)$. Such manipulations have nice applications in solving geometry problems.

As a very important fact that you should know (but which is hard to prove), we have the following:

Fundamental Theorem of Algebra. For any non-constant complex polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad a_0, \ldots, a_n \in \mathbb{C},$$

we can write (uniquely up to permutations)

$$p(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n),$$

where $\alpha_1, \dots, \alpha_n$ are (not necessarily distinct) complex numbers called the *roots* of p (they are precisely the solutions to the equation $p(\alpha) = 0$).

Problem 1. (Conjugation preserves operations). Let z, w be complex numbers. Show that

$$\overline{z+w} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{z} \cdot \overline{w}, \qquad \overline{z/w} = \overline{z}/\overline{w},$$

where in the division above we assume $w \neq 0$.

Problem 2. (Roots of unity). Let n be a positive integer.

(a) Find all complex solutions to the equation $z^n = 1$ (these are called the nth roots of unity).

(b) Show that there is a complex number ζ such that (formally)

$$x^n - 1 = (x-1)(x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1}).$$

(c) Show that for $n \geq 2$, the sum of all nth roots of unity found in part (a) is 0.

Problem 3. (Powers of complex numbers). Let n be a positive integer.

(a) Compute $(1+i)^n$ and $(1+i\sqrt{3})^n$.

(b) Using part (a), compute

$$\sum_{\substack{0 \leq k \leq n \\ k \text{ even}}} \binom{n}{k} (-1)^{k/2} = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots .$$

Problem 4. (Parallelism and perpendicularity). Let a, b, c, d be complex numbers such that $a \neq b$ and $c \neq d$.

(a) Show that the line through a and b is parallel or equal to the line through c and d if and only if

$$\frac{a-b}{c-d} = \frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}.$$

(Take $d = b$ to get a condition for when a, b, c are collinear.)

(b) Show that the line through a and b is perpendicular to the line through c and d if and only if

$$\frac{a-b}{c-d} = -\frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}.$$

Problem 5. (Midpoint and centroid). Let a, b, c be distinct complex numbers.

(a) Show that midpoint of the segment joining a and b is $\frac{a+b}{2}$.

(b) Show that the center of mass (centroid) of the triangle with vertices a, b, c is $\frac{a+b+c}{3}$.

Problem 6. (Orthocenter).

(a) Let a, b, c be distinct complex numbers on a circle centered at 0. Show that the orthocenter of the triangle with vertices a, b, c is given by $h = a + b + c$.

(b) In a non-equilateral triangle $\triangle ABC$, let O be the circumcenter, G the center of mass and H the orthocenter. Show that O, G and H are collinear in this order, and moreover that $GH = 2OG$.

Problem *7. (Equilateral triangles). Let a, b, c be distinct complex numbers. Show that the triangle with vertices a, b, c is *equilateral* if and only if

$$a^2 + b^2 + c^2 = ab + bc + ca.$$