INTER-CLASS COMPETITION PROBLEMS

LOS ANGELES MATH CIRCLE
ADVANCED 2

Problem 1 (Lehigh math competition 2007)**.**
What is the sum of the digits of $10^{55} - 55$?

Solution: $10^{55} - 55 = 99 \cdots 9945$ where there are 53 nines in that number. So the sum of the digits is $53 \cdot 9 + 9 = 486$.

Problem 2 (Lehigh math competition 2007)**.**
Find (with proof) all primes p such that $p^2 + 3p - 1$ is also prime.

Solution: If $p = 3$, then $3^2 + 3 \cdot 3 - 1 = 17$ which is prime. If $p \neq 3$, then $p^2 \equiv 1 \mod 3$, so $p^2 + 3p - 1 \equiv 0 \mod 3$ also. Since $p^2 + 3p - 1$ is never equal to 3, it can never be prime. So $p = 3$ is the only solution.

Problem 3 (Lehigh math competition 2007)**.**
What is the ratio of the area of a regular 10-gon to that of a regular 20-gon inscribed in the same circle? Express your answer using a single trig function with its angle in degrees.

Solution: Let the radius of the circle equal 1. The 10-gon consists of 20 right triangles with hypotenuse 1 and an angle of 18 degrees. Thus the area of the 10-gon is $20 \cdot \frac{1}{2} \sin(18) \cos(18) = 10 \sin(18) \cos(18)$. Similarly the area of the 20-gon is $20 \sin(9) \cos(9)$. By the double angle formula, this latter equals $10 \sin(18)$. Thus the ratio is $\boxed{\cos(18^\circ)}$.

Problem 4 (Lehigh math competition 2007)**.**
Let $P(n)$ denote the product of the digits of n, and let $S(n)$ denote the sum of the digits of n. Find (with proof) all positive integers satisfying $n = P(n) + S(n)$.

Solution: If n has one digit then $n = S(n) = P(n)$, so it can't satisfy the required equation. If $n = ab$ has two digits, then the required equation reads $10a + b = ab + a + b$, which implies $10a = (b + 1)a$, so $b = 9$ because $a \neq 0$. Then we can check that the numbers $19, 29, \ldots, 99$ are all solutions. If $n = abc$ has three digits, the required equation reads $100a + 10b + c = abc + a + b + c$, so $99a + 9b = abc$. But the left hand side here is $\geq 99a + 9$, and the right hand side is $\leq 81a$. Since $99a + 9 > 81a$ for all $1 \leq a \leq 9$, this

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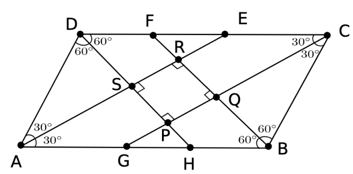
equation has no solutions. A similar argument will show that there are also no solutions with more than three digits. So the only solutions are $\boxed{19, 29, \ldots, 99}$.

Problem 5 (Georgia Tech high school group competition, 2018)**.**
Find (with proof) all $n \geq 2$ such that $\log_n(n+1)$ is rational.

Solution: There are no such n. If $\log_n(n+1) = p/q$ where p, q are positive integers, then $n^{p/q} = n+1$, so $n^p = (n+1)^q$. But we know that raising an integer to any integer power preserves the parity (an even number raised to any power is still even, and an odd number raises to any power is still odd). Thus n^p and $(n+1)^q$ must have opposite parities, a contradiction.

Problem 6 (Georgia Tech high school group competition, 2017)**.**
Suppose $ABCD$ is a parallelogram with side lengths $AB = 3, AD = 2.$ and $\angle DAB = 60°$. Find the area of the parallelogram formed by the intersections of the internal bisectors of of $\angle DAB, \angle ABC, \angle BCD, \angle CDA$.

Solution: 

Let E, F be points on CD, and G, H be points on AB such that AE, BF, CG and GH are the internal bisectors, as shown. Suppose they intersect in the points P, Q, R and S as shown. By angle chasing we see that $PQRS$ is a rectangle. We observe that $AH = BG = DE = CF = 2 = AD = BC$, and $AG = BH = CE = DF = 1$. By similarity, $SP/SH = AG/AH = 1/2$, and thus $SP = (1/2)SH = (1/2)AH\sin(30\circ) = 1/2$. By similarity, $PQ/GQ = HB/GB = 1/2$, and thus $PQ = (1/2)GQ = (1/2)GB\sin(60°) = \sqrt{3}/2$. Consequently, area of parallelogram $PQRS$ is $\sqrt{3}/4$.

Problem 7 (Georgia Tech high school proof competition, 2015)**.**
2015 points are chosen in a $7 \times 7 \times 15$ rectangular prism. Prove that the distance between some two points is less than or equal to 1.

Solution: Assume to the contrary that no such pair exists. Then we have a collection of 2015 disjoint balls of radius 1/2, centered at the given points, contained within the enlarged $8 \times 8 \times 16$ rectangular prism. The total volume of the balls is $2015(4/3)\pi(1/2)^3 = 2015\pi/6$, which is greater than the total available volume of 1024. This is a contradiction.

Problem 8 (Georgia Tecch high school proof competition, 2013)**.**
Prove or disprove: there exists a function f with domain and range equal to $\mathbb{R}$ such that the equation $f(x) = x$ has exactly one distinct solution and the equation $f(f(x)) = x$ has exactly two distinct solutions.

Solution: Let A be the set of a so that $f(a) = a$ and let B be the set of b such that $f(f(b)) = b$. Then $A \subseteq B$. Let $B' = B \setminus A$. We claim that if B' is finite, then it has even cardinality – if $b \in B'$, then $f(b)$ is also in B' and is not equal to b. Thus B' is partitioned into pairs $\{b, f(b)\}$. To finish, A is assumed to have cardinality 1, so B, if finite, has odd cardinality. Therefore no such function exists.

Alternate Solution: Suppose such a function exists. Let a be the unique solution to $f(x) = x$ (that is, $f(a) = a$). Then $f(f(a)) = f(a) = a$, so a is also a solution to $f(f(x)) = x$. Let the other distinct solution to this second equation be $b \neq a$. Then, $f(f(f(b))) = f(b)$, so $f(b)$ is also a solution to $f(f(x)) = x$. Since $f(f(x)) = x$ has only two distinct solutions, we must have either $f(b) = b$ or $f(b) = a$. If $f(b) = b$, then b is a second distinct solution to $f(x) = x$, a contradiction with our assumption. If $f(b) = a$, then $f(f(b)) = f(a) = a$, also a contradiction. Therefore no such function exists.

Problem 9 (Stanford Math Competition, team section, 2019)**.**
Find the maximum possible value of $\left|\sqrt{n^2+4n+5} - \sqrt{n^2+2n+5}\right|$ over all $n \in \mathbb{Z}$.

Solution: Notice that

$$\left|\sqrt{n^2+4n+5} - \sqrt{n^2+2n+5}\right| = \left|\sqrt{(n+2)^2+(0-1)^2} - \sqrt{(n+1)^2+(0-2)^2}\right|$$

If we let P be the point $(n, 0)$, A be the point $(-2, 1)$ and B be the point $(-1, 2)$ on the xy coordinate plane, then the expression above represents the difference between the lengths of PA and PB. By the triangle inequality, $\sqrt{2} = AB \geq |PA - PB|$, so $\sqrt{2}$ is an upper bound. This bound is achieved when A, B, P are collinear, which happens with $n = -3$. So the answer is $\sqrt{2}$.

Problem 10 (Stanford Math Competition, algebra section, 2018)**.** Given that the roots of the polynomial $x^3 - 7x^2 + 13x - 7$ are the real numbers r, s, t, compute the value of $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$.

Solution: The polynomial can be written as $x^2 - 7x^2 + 13x - 7 = (x-r)(x-s)(x-t)$, and by matching terms we find $rst = 7$ and $rs + rt + st = 13$. Therefore

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{st + rt + rs}{rst} = \frac{13}{7}.$$

Problem 11 (Stanford Math Competition, discrete section, 2018)**.**
A sequence is defined as follows. Given a term a_n, we define the the next term a_{n+1} by

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ a_n - 1 & \text{if } a_n \text{ is odd} \end{cases}$$

The sequence terminates when $a_n = 1$. Let $P(x)$ be the number of terms in such a sequence with initial term x. For example, $P(7) = 5$ because its corresponding sequence is $7, 6, 3, 2, 1$ Evaluate $P\left(2^{2018} - 2018\right)$.

Solution: We convert the number into its binary equivalent. If the number ends with 0 (which means it is even), the next term has the same binary form with the 0 removed. If the number ends with 1 (which means it is odd), the next term has the same binary form with the last digit changed to 0. Therefore $P(x)$ is simply equal to $2A + B - 1$ where A is the number of 1s in p and B is the number of 0s in p. Now we notice that

$$2^{2018} - 2018 \;=\; 2^{2018} - 1 - 2017 \;=\; 2^{2018} - 1 - (1 + 32 + 64 + 128 + 256 + 512 + 1024),$$

so in binary it has 2011 1s and seven 0s, so $P(2^{2018} - 2018) = 2 \cdot 2011 + 6 = 4028$.