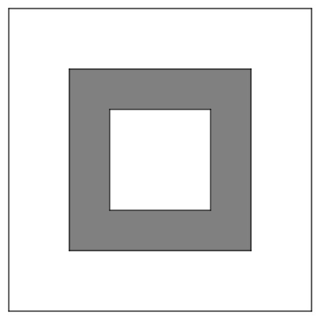
# OLYMPIAD-STYLE PROBLEMS II

COLLECTED FOR THE
LOS ANGELES MATH CIRCLE

**Problem 1** (2010 AIME II Problem 2 ©MAA)**.**
A point $P$ is chosen at random in the interior of a unit square $S$. Let $d(P)$ denote the distance from $P$ to the closest side of $S$. Find the probability that $1/5 \leq d(P) \leq 1/3$.

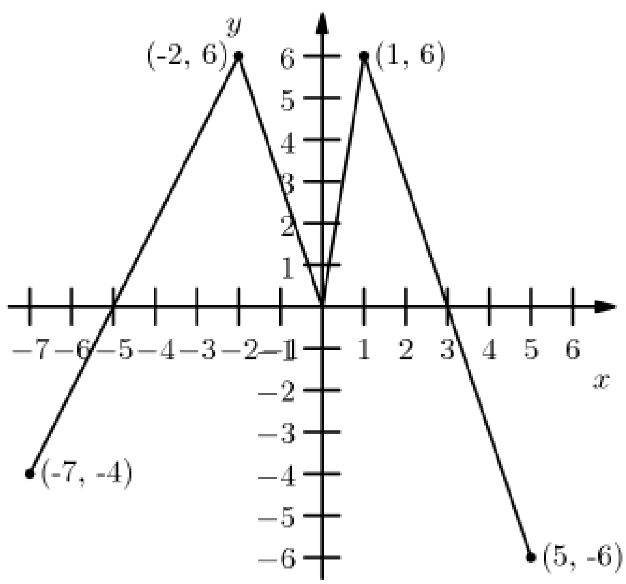
**Solution:** The event in question corresponds to the shaded region in the following picture.



The innermost square has vertices $(1/3, 1/3), (1/3, 2/3), (2/3, 2/3)$, and $(2/3, 1/3)$. The middle square has vertices $(1/5, 1/5), (1/5, 4/5), (4/5, 4/5)$, and $(4/5, 1/5)$. The outer square has vertices $(0, 0), (0, 1), (1, 1)$, and $(1, 0)$. Since the point $P$ is picked uniformly, the probability in question is just the area of the shaded region, which is

$$\left(\frac{3}{5}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{56}{225}.$$

**Problem 2** (2002 AMC 12A ©MAA)**.**
If $f : [-7, 5] \to \mathbb{R}$ is the function whose graph is shown below, how many solutions does the equation $f(f(x)) = 6$ have?

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**Solution:** Letting $y = f(x)$, we see that $f(f(x)) = 6$ if and only if $f(y) = 6$, which according to the graph occurs exactly when $y \in \{-2, 1\}$. Hence $f(f(x)) = 6$ if and only if $f(x) \in \{-2, 1\}$. Drawing horizontal lines on the graph at $y = -2$ and $y = 1$, we see that $f(x) = -2$ for two values of $x$ and $f(x) = 1$ for four values of $x$. Naturally since $-2 \neq 1$, these $x$-values must be distinct from each other, so in total we have 6 solutions to $f(f(x)) = 6$.

**Problem 3** (2006 AIME I Problem 3 ©MAA)**.**
Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

**Solution:** Suppose the original number is $N = \overline{a_n a_{n-1} \ldots a_1 a_0}$, where the $a_i$ are digits and the first digit, $a_n$, is nonzero. Then the number we create is $N_0 = \overline{a_{n-1} \ldots a_1 a_0}$, so

$$N = 29N_0.$$

But $N$ is $N_0$ with the digit $a_n$ added to the left, so $N = N_0 + a_n \cdot 10^n$. Thus,

$$N_0 + a_n \cdot 10^n = 29N_0$$
$$a_n \cdot 10^n = 28N_0.$$

The right-hand side of this equation is divisible by seven, so the left-hand side must also be divisible by seven. The number $10^n$ is never divisible by 7, so $a_n$ must be divisible by 7. But $a_n$ is a nonzero digit, so the only possibility is $a_n = 7$. This gives

$$7 \cdot 10^n = 28N_0$$

or

$$10^n = 4N_0.$$

Now, we want to minimize both $n$ and $N_0$, so we take $N_0 = 25$ and $n = 2$. Then

$$N = 7 \cdot 10^2 + 25 = 725,$$

and indeed, $725 = 29 \cdot 25$.

**Problem 4** (1988 AIME ©MAA)**.**
Suppose there is a function $f$ defined on the set of ordered pairs $(x, y)$ of positive integers which satisfies

$$f(x, x) = x, \tag{1}$$
$$f(x, y) = f(y, x), \quad \text{and} \tag{2}$$
$$(x + y)f(x, y) = yf(x, x + y). \tag{3}$$

Show that there is only one possible value of $f(14, 52)$ and find it.

**Solution:** The point here is that we can use the given identities to express $f(14, 52)$ in terms of something of the form $f(x, x)$, whose value we are given is $x$. Since our goal is to make the arguments of the function equal, it makes sense to try to do so by making them both small; after all, there are only $n$ positive integers less than or equal to $n$, so our chances of finding overlap are good for small $n$. To do so, first note that the third given equation allows us to replace one argument by the difference between the two (provided it remains positive) since

$$(x + y - x)f(x, y - x) = (y - x)f(x, y - x + x) \implies f(x, y) = \frac{y}{y - x} f(x, y - x). \tag{4}$$

Using this, together with the second given equation,

$$\begin{aligned} f(14,52) &= \frac{52}{38} f(14,38) = \frac{52}{38}\cdot\frac{38}{24} f(14,24) = \frac{52}{38}\cdot\frac{38}{24}\cdot\frac{24}{10} f(14,10) \\ &= \frac{52}{10} f(10,14) = \frac{52}{10}\cdot\frac{14}{4} f(10,4) \\ &= \frac{13\cdot 7}{5} f(4,10) = \frac{13\cdot 7}{5}\cdot\frac{10}{6} f(4,6) = \frac{13\cdot 7}{5}\cdot\frac{10}{6}\cdot\frac{6}{2} f(4,2) \\ &= 13\cdot 7\cdot f(2,4) = 13\cdot 7\cdot \frac{4}{2} f(2,2) \\ &= 13\cdot 7\cdot 2\cdot 2 = 364. \end{aligned}$$

We remark that there is such a function, namely $f(x,y) = \mathrm{lcm}(x,y)$, and indeed, 364 is the least common multiple of 14 and 52.

**Problem 5.** Consider a circle with diameter AB. Let C be a point outside of the circle. Suppose AC and BC intersect the circle at points D and M respectively. The areas of triangle DCM is 1/4 of the area of triangle ACB. Find angle CBD.

**Hint:** Triangles BCA and DCM are similar. The ratio of sides is 2. BC=2CD. Since sin (angle CBD)=1/2 and the angle is acute, it equals to 30 degrees.

**Problem 6** (2008 AIME I Problem 11 ©MAA)**.**
Consider sequences that consist entirely of $A$'s and $B$'s and that have the property that every run of consecutive $A$'s has even length, and every run of consecutive $B$'s has odd length. Examples of such sequences are $AA$, $B$, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

**Solution:** (AoPS) Let $a_n$ and $b_n$ denote, respectively, the number of sequences of length $n$ ending in $A$ and $B$. If a sequence ends in an $A$, then it must have been formed by appending two $A$s to the end of a string of length $n-2$. If a sequence ends in a $B$, it must have either been formed by appending one $B$ to a string of length $n-1$ ending in an $A$, or by appending two $B$s to a string of length $n-2$ ending in a $B$. Thus, we have the recursions

$$a_n = a_{n-2} + b_{n-2}$$
$$b_n = a_{n-1} + b_{n-2}$$

By counting, we find that $a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 0$.

| $n$ | $a_n$ | $b_n$ | $n$ | $a_n$ | $b_n$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 8 | 6 | 10 |
| 2 | 1 | 0 | 9 | 11 | 11 |
| 3 | 1 | 2 | 10 | 16 | 21 |
| 4 | 1 | 1 | 11 | 22 | 27 |
| 5 | 3 | 3 | 12 | 37 | 43 |
| 6 | 2 | 4 | 13 | 49 | 64 |
| 7 | 6 | 5 | 14 | 80 | 92 |

Therefore, the number of such strings of length 14 is $a_{14} + b_{14} = \boxed{172}$.

**Problem 7** (2004 Manhattan Mathematical Olympiad)**.**
Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.

**Solution:** Note $a, b, c \in [1, 10]$ can be side lengths of a triangle if and only if $|a - b| < c, |a - c| < b$ and $|b - c| < a$.

Notice if $a, b, c \in [k, 2k)$ for some $k \geq 1$, then $|a - b| < 2k - k = k \leq c$, and the other inequalities are similarly satisfied. So in this case, we may form a triangle.

On the other hand, we are choosing 7 points in the interval $[1, 10] = [1, 2) \cup [2, 4) \cup [4, 10]$. By the pigeonhole principle, one of the three subintervals must contain 3 chosen lengths. If there are three in $[1, 2)$ or $[2, 4)$, by the above case, we are done. Otherwise, we must have 3 of the chosen lengths in the interval $[4, 10]$. In fact, we must have two points in $[1, 2)$, two in $[2, 4)$ and three in $[4, 10]$ (otherwise, we would necessarily be in the previous case, again by pigeonhole). Since we have three points in $[4, 10] = [4, 7) \cup [7, 10]$, we know that by pigeonhole we either have two points in $[4, 7)$ or $[7, 10]$. Either way, we have two points $a, b \in [4, 10]$ with $|a - b| < 3$.

On the other hand, we have two points in $[2, 4) = [2, 3) \cup [3, 4)$. If they both fall in the first interval, we have two points $x, y \in [2, 4)$ with $|x - y| < 1$. In this case, take any chosen point $z \in [1, 2)$ and $x, y, z$ will satisfy the desired conditions. Otherwise, we must have at least one point $c$ in the interval $[3, 4)$. Then $a, b, c$ will satisfy the desired conditions.

**Problem 8** (2003 AIME I Problem 11 ©MAA)**.**
Let $0 \leq x \leq 90$ be chosen uniformly at random. What is the probability that the numbers $\sin^2 x$, $\cos^2 x$, and $\sin x \cos x$ do not form a triangle?

**Note:** $x$ is measured in degrees, and you may leave your answer in terms of the arctan() (inverse tangent) function.

**Solution:** (AoPS) First notice that the problem is symmetric about $x = 45°$ – if we replace $x$ by $90 - x$, then $\sin x$ becomes $\cos x$ and vice versa, so we are still dealing with the same three numbers $\sin^2 x$, $\cos^2 x$, and $\sin x \cos x$. Thus we may assume without loss of generality that $0 \leq x \leq 45$. In this case, $\cos^2 x \geq \sin x \cos x \geq \sin^2 x$, so the triangle inequality says that a triangle cannot be formed if and only if $\sin^2 x + \sin x \cos x \leq \cos^2 x$. Rewrite this as

$$\frac{1}{2}\sin(2x) \;=\; \sin x \cos x \;\leq\; \sin^2 x - \cos^2 x \;=\; \cos(2x).$$

Since $0 \leq x \leq 45$, both $\sin(2x)$ and $\cos(2x)$ are $\geq 0$, so this happens if and only if

$$\tan(2x) \;\leq\; 2 \iff x \;\leq\; \frac{1}{2}\arctan(2).$$

Since $x$ is picked uniformly between 0 and 45, the probability of $x \leq \frac{1}{2}\arctan(2)$ is

$$\frac{1}{45} \cdot \frac{\arctan(2)}{2} \;\approx\; 0.70.$$

**Problem 9** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.**
Prove that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$$

Generalize those facts to

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \dots$$
$$\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \binom{n}{5}\cos^{n-5}\theta\sin^5\theta - \dots$$

**Hint:** It may be helpful to recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

**Solution:** We start by expressing $\cos n\theta$ and $\sin n\theta$ as the real and imaginary parts of $e^{in\theta} = (e^{i\theta})^n$. Then expressing $e^{i\theta} = \cos\theta + i\sin\theta$, we expand with the binomial theorem, and are now looking for the real and imaginary parts of $\sum_{k=0}^{n} i^k \binom{n}{k} \cos^{n-k}\theta \sin^k\theta$. The real terms are those where $k$ is even, and the imaginary terms are those where $k$ is odd, so replacing $k$ with $k/2$, we get $\cos n\theta = \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n}{2k} \cos^{n-2k}\theta \sin^{2k}\theta$, and replacing $k$ with $(k-1)/2$, we get $\sin n\theta = \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}\theta \sin^{2k+1}\theta$.

**Problem 10** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.**
Using the previous problem, evaluate the following expressions:

$$\cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + \cdots + \cot^2\frac{n\pi}{2n+1}$$
$$\csc^2\frac{\pi}{2n+1} + \csc^2\frac{2\pi}{2n+1} + \cdots + \csc^2\frac{n\pi}{2n+1}$$

**Hint:** Using the previous problem's answer, find a polynomial with solutions $\cot^2\frac{k\pi}{2n+1}$ for $k = 1, \ldots, n$. Then recall that you can easily determine the sum of the roots of a polynomial from its coefficients.

**Solution:** Start by replacing $n$ in the previous problem with $2n+1$, getting $\sin((2n+1)\theta) = \sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}\cos^{2n-2k}\theta\sin^{2k+1}\theta$. Then divide both sides by $\sin^{2n+1}\theta$, getting

$$\frac{\sin((2n+1)\theta)}{\sin^{2n+1}\theta} = \sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}\cot^{2(n-k)}\theta.$$

Then we see that if we let $\theta = \frac{k\pi}{2n+1}$ for $1 \le k \le n$, the left side is 0, so $\cot^2\theta$ is a solution of the polynomial equation $\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}x^{n-k}$. As this is a degree $n$ polynomial, and we have found $n$ distinct solutions, it has no more.

Now we recall that the sum of the roots of the degree $n$ polynomial $p(x) = \sum_{k=0}^{n} a_k x^n$ is $-\frac{a_{n-1}}{a_n}$, which in this case is $\frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{n(2n-1)}{3}$, which gives us the value of the first expression.

Then we use the identity $\cot^2\theta + 1 = \csc^2\theta$, showing us that the second sum is just $n$ greater, as it has $n$ terms, with value $\frac{2n(n+1)}{3}$.

**Problem 11** (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.**
Using the last problem, show that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ lies between

$$\left(1 - \frac{1}{2n+1}\right)\left(1 - \frac{2}{2n+1}\right)\frac{\pi^2}{6} \quad\text{and}\quad \left(1 - \frac{1}{2n+1}\right)\left(1 + \frac{1}{2n+1}\right)\frac{\pi^2}{6}$$

**Hint:** Use (and prove) the fact that if $0 < \theta < \frac{\pi}{2}$, then $\sin\theta < \theta < \tan\theta$.

**Solution:** The inequalities in the hint are best proved by drawing a picture.

We can invert the inequalities to get $\csc\theta < \frac{1}{\theta} < \cot\theta$.

**Problem 12** (2006 AIME I Problem 13 ©MAA)**.**
For each even positive integer $x$, let $g(x)$ denote the greatest power of 2 that divides $x$. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer $n$, let $S_n = \sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer $n$ less than 1000 such that $S_n$ is a perfect square.

**Solution:** Given $g : x \mapsto \max_{j:2^j|x} 2^j$, consider $S_n = g(2) + \cdots + g(2^n)$. Define $S = \{2, 4, \ldots, 2^n\}$. There are $2^0$ elements of $S$ that are divisible by $2^n$, $2^1 - 2^0 = 2^0$ elements of $S$ that are divisible by $2^{n-1}$ but not by $2^n, \ldots,$ and $2^{n-1} - 2^{n-2} = 2^{n-2}$ elements of $S$ that are divisible by $2^1$ but not by $2^2$.

Thus

$$\begin{aligned} S_n &= 2^0 \cdot 2^n + 2^0 \cdot 2^{n-1} + 2^1 \cdot 2^{n-2} + \cdots + 2^{n-2} \cdot 2^1 \\ &= 2^n + (n-1)2^{n-1} \\ &= 2^{n-1}(n+1). \end{aligned}$$

Let $2^k$ be the highest power of 2 that divides $n+1$. Thus by the above formula, the highest power of 2 that divides $S_n$ is $2^{k+n-1}$. For $S_n$ to be a perfect square, $k+n-1$ must be even. If $k$ is odd, then $n+1$ is even, hence $k+n-1$ is odd, and $S_n$ cannot be a perfect square. Hence $k$ must be even. In particular, as $n < 1000$, we have five choices for $k$, namely $k = 0, 2, 4, 6, 8$.

If $k = 0$, then $n+1$ is odd, so $k+n-1$ is odd, hence the largest power of 2 dividing $S_n$ has an odd exponent, so $S_n$ is not a perfect square.

In the other cases, note that $k+n-1$ is even, so the highest power of 2 dividing $S_n$ will be a perfect square. In particular, $S_n$ will be a perfect square if and only if $(n+1)/2^k$ is an odd perfect square.

If $k = 2$, then $n < 1000$ implies that $\frac{n+1}{4} \leq 250$, so we have $n+1 = 4, 4 \cdot 3^2, \ldots, 4 \cdot 13^2, 4 \cdot 3^2 \cdot 5^2$.

If $k = 4$, then $n < 1000$ implies that $\frac{n+1}{16} \leq 62$, so $n+1 = 16, 16 \cdot 3^2, 16 \cdot 5^2, 16 \cdot 7^2$.

If $k = 6$, then $n < 1000$ implies that $\frac{n+1}{64} \leq 15$, so $n+1 = 64, 64 \cdot 3^2$.

If $k = 8$, then $n < 1000$ implies that $\frac{n+1}{256} \leq 3$, so $n+1 = 256$.

Comparing the largest term in each case, we find that the maximum possible $n$ such that $S_n$ is a perfect square is $4 \cdot 3^2 \cdot 5^2 - 1 = 899$.

**Problem 13.** Consider circumscribed circle of triangle ABC. For any point M on the circle, consider it's projections P and Q on the sides AC and BC respectively. Find point M so that the length of PQ is as big as possible.

**Solution:** Points M,P,Q,C lie on a circle (not the circumcircle, of course!). Thus, PQ=MC sin (angle ACB). Therefore, PQ is maximal if MC is maximal, i.e., MC is the diameter of the circumcircle going through point C.

**Problem 14** (2016 Putnam Problem B4 ©MAA)**.**
Let $A$ be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of $n$), where $A^t$ is the transpose of $A$.

**Solution:** (From Kiran Kedlaya's archive) Recall the following determinant formula. For any $d \times d$ matrix $B = (B_{i,j})$,

$$\det(B) \;=\; \sum_{\sigma \in \mathrm{Perm}(d)} \mathrm{sgn}(\sigma) \prod_{i=1}^{d} B_{i,\sigma(i)}$$

where Perm($d$) is the set of all permutations of $\{1,\ldots,d\}$ and $\mathrm{sgn}(\sigma)=1$ if $\sigma$ is an even permutation and $-1$ is $\sigma$ is an odd permutation. Apply this formula to $A-A^t$ and get

$$\begin{aligned}\det(A-A^t) &= \sum_{\sigma\in\mathrm{Perm}(2n)}\mathrm{sgn}(\sigma)\prod_{i=1}^{2n}(A-A^t)_{i,\sigma(i)}\\ &= \sum_{\sigma\in\mathrm{Perm}(2n)}\mathrm{sgn}(\sigma)\prod_{i=1}^{2n}\left(A_{i,\sigma(i)}-A_{\sigma(i),i}\right) \;=:\; \sum_{\sigma\in\mathrm{Perm}(2n)}\mathrm{sgn}(\sigma)\cdot E_\sigma,\end{aligned}$$

so by linearity of expectation

$$\mathbb{E}(\det(A-A^t)) \;=\; \sum_{\sigma\in\mathrm{Perm}(2n)}\mathrm{sgn}(\sigma)\cdot\mathbb{E}(E_\sigma).$$

Let us examine $E_\sigma=\prod_{i=1}^{2n}(A_{i,\sigma(i)}-A_{\sigma(i),i})$. It is easiest to see what is going on by loooking at an example. Suppose $n=3$ and $\sigma=(14)(2)(356)$ in disjoint cycle notation. Then

$$\begin{aligned}E_\sigma &= (A_{1,4}-A_{4,1})(A_{2,2}-A_{2,2})(A_{3,5}-A_{5,3})(A_{4,1}-A_{1.4})(A_{5,6}-A_{6,5})(A_{6,3}-A_{3,6})\\ &= (A_{1,4}-A_{4,1})(A_{4,1}-A_{1,4})\cdot(A_{2,2}-A_{2,2})\cdot(A_{3,5}-A_{5,3})(A_{5,6}-A_{6,5})(A_{6,3}-A_{3,6}).\end{aligned}$$

Since all of the $A_{i,j}$ are independent, the expectation splits up as

$$\begin{aligned}&\mathbb{E}(E_\sigma)\\ &= \mathbb{E}\Big((A_{1,4}-A_{4,1})(A_{4,1}-A_{1,4})\Big)\cdot\mathbb{E}\Big((A_{2,2}-A_{2,2})\Big)\cdot\mathbb{E}\Big((A_{3,5}-A_{5,3})(A_{5,6}-A_{6,5})(A_{6,3}-A_{3,6})\Big)\\ &= \mathbb{E}\Big((A_{1,4}-A_{4,1})(A_{4,1}-A_{1,4})\Big)\cdot\mathbb{E}(A_{2,2}-A_{2,2})\cdot\mathbb{E}(A_{3,5}-A_{5,3})\cdot\mathbb{E}(A_{5,6}-A_{6,5})\cdot\mathbb{E}(A_{6,3}-A_{3,6}).\end{aligned}$$

Now since each $A_{i,j}$ has the same distribution (0 with probability 1/2 and 1 with probability 1/2), all of the expectations in the above product (except for the first one) are equal to $1/2-1/2=0$, so we get 0 for the entire product, so $\mathbb{E}(E_\sigma)=0$.

However, if $\sigma$ had the form of something like (12)(34)(56) in disjoint cycle notation, then *all* of the terms in the expression for $\mathbb{E}(E_\sigma)$ would have the same form as the first term above, and therefore would not be equal to 0. So, returning to the general case, we see that the only elements $\sigma\in\mathrm{Perm}(2n)$ that do not have $\mathbb{E}(E_\sigma)=0$ are those $\sigma$ of the form $(i_1i_2)(i_3i_4)\cdots(i_{2n-1}i_{2n})$ in disjoint cycle notation. Let $S_{2n}$ denote the set of all $\sigma\in\mathrm{Perm}(2n)$ of this form. Then our original expression for $\mathbb{E}(\det(A-A^t))$ reduces to

$$\mathbb{E}(\det(A-A^t)) \;=\; (-1)^n\sum_{\sigma\in S_{2n}}\mathbb{E}(E_\sigma)$$

because each $\sigma\in S_{2n}$ has sgn equal to $(-1)^n$. Now, for $\sigma=(i_1i_2)(i_3i_4)\cdots(i_{2n-1}i_{2n})\in S_n$, we can calculate

$$\begin{aligned}\mathbb{E}(E_\sigma) &= \mathbb{E}\Big((A_{i_1,i_2}-A_{i_2,i_1})(A_{i_2,i_1}-A_{i_1,i_2})\Big)\cdots\mathbb{E}\Big((A_{i_{2n-1},i_{2n}}-A_{i_{2n},i_{2n-1}})(A_{i_{2n},i_{2n-1}}-A_{i_{2n-1},i_{2n}})\Big)\\ &= \mathbb{E}(2A_{i_1.i_2}A_{i_2,i_1}-A_{i_1,i_2}^2-A_{i_2,i_1}^2)\cdots\mathbb{E}(2A_{i_{2n-1}.i_{2n}}A_{i_{2n},i_{2n-1}}-A_{i_{2n-1},i_{2n}}^2-A_{i_{2n},i_{2n-1}}^2)\\ &= (2(1/2)(1/2)-1/2-1/2)\cdots(2(1/2)(1/2)-1/2-1/2) \;=\; (1/2)^n(-1)^n.\end{aligned}$$

Therefore we have

$$\mathbb{E}(\det(A-A^t)) \;=\; (-1)^n\sum_{\sigma\in S_{2n}}\mathbb{E}(E_\sigma) \;=\; (-1)^n\sum_{\sigma\in S_{2n}}2^{-n}(-1)^n \;=\; \frac{|S_{2n}|}{2^n}.$$

Now we must just count the number of $\sigma\in\mathrm{Perm}(2n)$ of the form $(i_1i_2)(i_3i_4)\cdots(i_{2n-1}i_{2n})$. This number is simply equal to the number of ways to split a group of $2n$ objects into $n$ groups of 2, which is $\frac{(2n)!}{n!2^n}$, and therefore the final answer is

$$\mathbb{E}(\det(A-A^t)) \;=\; \frac{(2n)!}{n!4^n}.$$