OLYMPIAD-STYLE PROBLEMS I

COLLECTED FOR THE

LOS ANGELES MATH CIRCLE

Problem 1 (USAMTS Year 18 - Round 1 - Problem 4 ©Art of Problem Solving Initiative)**.** Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.

Solution: (AoPS) Consider a 4×82 rectangle of points in the plane, such as $\{(x, y) \in \mathbb{Z}^2 : 0 \le x \le 3, 0 \le y \le 81\}$. For each column, there are 4 points and 3 possible colors per point, for a total of $3^4 = 81$ possible colorings. With 82 columns, by the Pigeonhole Principle, there are two columns with the same coloring. Also, there are 4 points per column and 3 possible colors, so by the Pigeonhole Principle, some color appears twice. From each of the two columns, take some corresponding two points of a color that appears twice. These form a rectangle all of whose vertices are the same color.

Problem 2. Can you draw a path on the surface of Rubik's cube (3x3x3 cube) that goes through every single square on the surface? The path should not go through any vertices.

The answer is yes. They can draw an example on a net of the cube.

Problem 3 (1983 AIME ©MAA)**.**
Find the product of all real solutions to $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$.

Solution: If x satisfies the given equation, then $y = x^2 + 18x + 30$ has $y = 2\sqrt{y + 15}$. Hence $y^2 = 4y + 60$, factoring to $(y - 10)(y + 6) = 0$. We cannot have $y = -6$, since $y = 2\sqrt{y + 15}$ is non-negative, so we must have $y = 10$.

Thus $x^2 + 18x + 30 = y = 10$ so that $x^2 + 18x + 20 = 0$. Since the coefficient on x^2 is 1, we get that the roots of this quadratic sum to -18 and have product 20. Since these roots are exactly the solutions to the given equation, the answer is 20.

Problem 4. Let C be a circle with center O. Let A be a point inside of the circle. Consider the set midpoints of all possible chords going through point A. Describe this set (with proof).

Hint: Draw a chord BC going through pt A. Let M be the midpoint of the chord. Triangle AMO is right angle triangle. The set of all such points M forms the circle for which OA is the diameter. Make sure students fill in the details when submitting solutions. In particular, they need to show that if you take any point on the circle there is a chord for which this point is the midpoint.

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Problem 5 (2004 AIME II Problem 8 ©MAA)**.**
How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

The prime factorization of 2004 is $2^2 \cdot 3 \cdot 167$. Thus the prime factorization of 2004^{2004} is $2^{4008} \cdot 3^{2004} \cdot 167^{2004}$.

We can count the number of divisors of a number by multiplying together one more than each of the exponents of the prime factors in its prime factorization. For example, the number of divisors of $2004 = 2^2 \cdot 3^1 \cdot 167^1$ is $(2+1)(1+1)(1+1) = 12$.

A positive integer divisor of 2004^{2004} will be of the form $2^a \cdot 3^b \cdot 167^c$. Thus we need to find how many (a, b, c) satisfy

$$(a+1)(b+1)(c+1) = 2^2 \cdot 3 \cdot 167.$$

We can think of this as partitioning the exponents to $a+1$, $b+1$, and $c+1$. So let's partition the 2's first. There are two 2's so this is equivalent to partitioning two items in three containers. We can do this in $\binom{4}{2} = 6$ ways. We can partition the 3 in three ways and likewise we can partition the 167 in three ways. So we have $6 \cdot 3 \cdot 3 = 54$ as our answer.

Problem 6 (2002 AMC 12A Problem 16 ©MAA)**.**
Tina randomly selects two distinct numbers from $\{1, 2, 3, 4, 5\}$. Sergio randomly selects one number from the set $\{1, 2, \ldots, 10\}$. What is the probability that Sergio's number is greater than the sum of the two numbers chosen by Tina?

Solution: Let Sergio's number be S and Tina's numbers be T_1 and T_2. First note there are $\binom{5}{2} = 10$ choices for the unordered pair $\{T_1, T_2\}$. If $S = 1$ or $S = 2$ or $S = 3$, then $S > T_1 + T_2$ is impossible (because T_1 and T_2 are distinct). If $S = 4$, there is only one acceptable pair – $\{1, 2\}$ – so the probability of success is $1/10$ (success being defined as $S > T_1 + T_2$). Continuing in this way, we have

S	# of acceptable pairs	Probability of success
5	2	2/10
6	4	4/10
7	6	6/10
8	8	8/10
9	9	9/10
10	10	1

Then the total proability of success is obtained by averaging all of these –

$$\text{probability of success} \;=\; \frac{1}{10}\left(0+0+0+\frac{1}{10}+\frac{2}{10}+\frac{4}{10}+\frac{6}{10}+\frac{8}{10}+\frac{9}{10}+1\right) \;=\; \frac{2}{5}.$$

Problem 7 (2006 AIME I Problem 11 ©MAA)**.**
A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \le k \le 8$. A tower is to be built using all 8 cubes according to the rules:

(1) Any cube may be the bottom cube in the tower.
(2) The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

Solution: (AoPS) We proceed recursively. Suppose we can build T_m towers using blocks of size $1, 2, \dots, m$. How many towers can we build using blocks of size $1, 2, \dots, m, m+1$? If we remove the block of size $m+1$ from such a tower (keeping all other blocks in order), we get a valid tower using blocks $1, 2, \dots, m$. Given a tower using blocks $1, 2, \dots, m$ (with $m \geq 2$), we can insert the block of size $m+1$ in exactly 3 places: at the beginning, immediately following the block of size $m-1$ or immediately following the block of size m. Thus, there are 3 times as many towers using blocks of size $1, 2, \dots, m, m+1$ as there are towers using only $1, 2, \dots, m$. There are 2 towers which use blocks $1, 2$, so there are $2 \cdot 3^6 = 1458$ towers using blocks $1, 2, \dots, 8$, so the answer is $\boxed{458}$.

Problem 8. Consider a square of size 102x102 drawn on grid paper. Consider a shape consisting of 101 grid squares total. The shape considered is connected (that is, for any two squares in the shape, there is a chain of squares sharing a side that starts with the first square and ends with the second. Diagonal connections (via corners) are not allowed).

What is the largest integer n such that we are guaranteed to be able to cut n copies of the given shape out of the square of 102x102 (independently of what the shape is)?

Solution:

1. any connected shape consisting of 101 squares can be fit into a rectangle with sides a, b such that a+b=102. (make sure they prove this!).

2. For any rectangle of size a x b with a+b=102, we can cut four such rectangles out of the square of size 102 x 102 (going around the border).

Thus, we can always cut 4 shapes (one per rectangle).

3. Consider the cross with each side being 25 squares long, and one center square (total number of squares is 25 x 4 +1= 101). You can show that you can not cut out more than 4 such crosses out of the square of size 102 x 102.

Thus, the answer is 4.

Problem 9 (2005 AIME I Problem 12 ©MAA)**.**
For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

It is well-known that $\tau(n)$ is odd if and only if n is a perfect square. (Otherwise, we can group divisors into pairs whose product is n.) Thus, $S(n)$ is odd if and only if there are an odd number of perfect squares less than n. So $S(1), S(2)$ and $S(3)$ are odd, while $S(4), S(5), \dots, S(8)$ are even, and $S(9), \dots, S(15)$ are odd, and so on.

So, for a given n, if we choose the positive integer m such that $m^2 \leq n < (m+1)^2$ we see that $S(n)$ has the same parity as m.

It follows that the numbers between 1^2 and 2^2, between 3^2 and 4^2, and so on, all the way up to the numbers between 43^2 and $44^2 = 1936$ have $S(n)$ odd. These are the only such numbers less than 2005 (because $45^2 = 2025 > 2005$).

Notice that the difference between consecutive squares are consecutively increasing odd numbers. Thus, there are 3 numbers between 1 (inclusive) and 4 (exclusive), 5 numbers between 4 and 9, and so on. The number of numbers from n^2 to $(n+1)^2$ is $(n+1-n)(n+1+n) = 2n+1$. Whenever the lowest square beneath a number is odd, the parity will be odd, and the same for even. Thus, $a = [2(1)+1] + [2(3)+1] \dots [2(43)+1] = 3 + 7 + 11 \dots 87$. $b = [2(2)+1] + [2(4)+1] \dots [2(42)+1] + 70 = 5 + 9 \dots 85 + 70$, the 70 accounting for the difference between 2005 and $44^2 = 1936$, inclusive. Notice that if we align the two and subtract, we get that each difference is equal to 2. Thus, the solution is $|a - b| = |b - a| = |2 \cdot 21 + 70 - 87| = 25$.

Problem 10 (1988 IMO, proposed by Stephan Beck)**.**
Suppose that a and b are positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer.
Show that k must be a perfect square.

Hint: This problem is a somewhat famous example of the power of the *method of infinite descent*, which focuses on contradicting the existence of a "minimal" example or counterexample. For instance, for this problem, suppose toward a contradiction that the result of the problem is false, and let S denote the (then nonempty) set of pairs (a, b) of positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer but not a perfect square.

We can measure the "size" of a given counterexample by the sum $a + b$ and since $\{a + b : (a, b) \in S\}$ is a set of positive integers, it contains a minimal element. That is, we can find a pair $(a, b) \in S$ with the property that $a + b \leq a' + b'$ for any $(a', b') \in S$. To finish the problem, produce a contradiction by producing a strictly smaller counterexample; that is, from this pair (a, b) find some $(a', b') \in S$ with $a' + b' < a + b$.

As an extra hint, we remark that if you relabel as necessary to ensure that $a \geq b$, you will even be able to take $b' = b$ and use the same integer k.

Solution: Let (a, b) be as in the hint, and following the extra hint, relabeling if necessary we assume without loss of generality that $a \geq b$.

Let $k = \frac{a^2+b^2}{ab+1}$, and note that this means that a is a root of the polynomial $p(x) = x^2 - kbx + b^2 - k$. Letting a' denote the other root[1] of p, since the coefficient on x^2 is 1, we know that the sum and product these roots satisfy

$$a + a' = bk, \quad \text{and} \tag{1}$$

$$aa' = b^2 - k. \tag{2}$$

From (1) we see that a' is an integer and from $a'^2 + b^2 = k(a'b + 1)$ we see that it must be non-negative (lest $a'b+1$ become non-positive). But we also cannot have $a' = 0$, since (2) would then give $k = b^2$, a perfect square. Thus a' is a positive integer with $k = \frac{a'^2+b^2}{a'b+1}$.

Finally, note that (2) gives

$$a' = \frac{b^2 - k}{a} \leq \frac{a^2 - k}{a} = a - \frac{k}{a} < a, \tag{3}$$

so that $a' + b < a + b$, the desired contradiction to the minimality of $a + b$.

[1] Allowing for the moment that this may be a repeat of a, although we will see that they must be distinct.

For the next two problems, it may be helpful to recall that $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.

Problem 11 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.** Show that

$$\cos \frac{2\pi}{2n+1} + \cos \frac{4\pi}{2n+1} + \cdots + \cos \frac{2n\pi}{2n+1} = -\frac{1}{2}.$$

The term $\cos \frac{2\pi k}{2n+1}$ is the real part of $\exp\left(\frac{2\pi k i}{2n+1}\right) = z^k$, where $z = \exp\left(\frac{2\pi i}{2n+1}\right)$. Thus the left side of our equation, which we will call L, is the real part of $\sum_{k=1}^{n} z^k$. As

$\cos(x) = \cos(2\pi - x)$ for all x, $\cos \frac{2\pi k}{2n+1} = \cos \frac{2\pi(2n+1-k)}{2n+1}$ for all k, so L also equals

$$\sum_{k=1}^{n} \cos \frac{2\pi(2n+1-k)}{2n+1} = \sum_{k=n+1}^{2n} z^k$$

Thus $2L = \sum_{k=1}^{2n} z^k$, and it suffices to show that $1 + \sum_{k=1}^{2n} z^k$ has real part 0. In fact we will show it is 0. First we note that $1 + \sum_{k=1}^{2n} z^k = \sum_{k=0}^{2n} z^k$, and then we multiply this polynomial in z by $z - 1$, and terms cancel, giving $z^{2n+1} - 1$. We find that $z^{2n+1} = \exp\left((2n+1)\frac{2\pi i}{2n+1}\right) = e^{2\pi i} = 1$, so $z^{2n+1} - 1 = 0$, but $z - 1 \neq 0$, so $\sum_{k=0}^{2n} z^k = \frac{z^{2n+1}-1}{z-1} = 0$.

Problem 12 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.**
Let $A_1A_2\dots A_n$ be a regular polygon circumscribed by a circle C of radius r and center O. If P is a point on that circle, show the sum of the squares of the distances from P to the vertices of the polygon is $2nr^2$.

Show that if instead P is not necessarily on the circle, that the sum of the squares of the distances from P to the vertices of the polygon is $n(r^2 + l^2)$, where l is the distance from P to O.

We may assume that $r = 1$, as scaling the problem up linearly will multiply the squares of the distances, and thus the answer, by r^2. Up to rotation and translation, we may assume that O is 0 of the complex plane, and P is at 1. Also assume that the points go counterclockwise around the circle, in order $A_1, A_2, \dots, A_n$. Then as A_k is on the unit circle, if it is at a counterclockwise angle θ_k around the circle from P, A_k is located at complex number $e^{i\theta_k} = \cos\theta_k + i \sin\theta_k$. We can describe the square of the distance between the points as $|1 - e^{i\theta_k}|^2 = (1 - e^{i\theta_k})\overline{(1 - e^{i\theta_k})}$. As $\cos(-\theta_k) = \cos(\theta_k)$ and $\sin(-\theta_k) = -\theta_k$, we see that $e^{-i\theta_k} = \overline{e^{i\theta_k}}$, so

$$(1 - e^{i\theta_k})\overline{(1 - e^{i\theta_k})} = (1 - e^{i\theta_k})(1 - e^{-i\theta_k}) = 2 - e^{i\theta_k} - e^{-i\theta_k} = 2 - 2\cos\theta_k.$$

Thus the sum of the squares of the distances is $\sum_{k=1}^{n}(2 - 2\cos\theta_k) = 2n - 2\sum_{k=1}^{n}\cos\theta_k$, and it suffices to show that $\sum_{k=1}^{n}\cos\theta_k = 0$. As $\sum_{k=1}^{n}\cos\theta_k$ is the real part of $\sum_{k=1}^{n} e^{i\theta_k}$, it will be enough to show that $\sum_{k=1}^{n} e^{i\theta_k} = 0$. We claim that rotating the problem again by angle φ will not change the answer, as this would just replace each θ_k with $\theta_k + \varphi$, and $\sum_{k=1}^{n} e^{i(\theta_k+\varphi)} = e^{i\varphi}\sum_{k=1}^{n} e^{i\theta_k}$, so if one number is 0, the other is as well. Thus we may assume that $\theta_n = 2\pi$, and that the other points are each $\frac{1}{n}$ of a full-circle rotation away from each other, counterclockwise, so that $\theta_k = \frac{2\pi k}{n}$. Thus our sum is $\sum_{k=1}^{n} e^{ik\theta}$, where $\theta = \frac{2\pi}{n}$, or if $z = e^{i\theta}$, $\sum_{k=1}^{n} z^k = z\sum_{k=0}^{n-1} z^k$, and this is 0, as multiplying $\sum_{k=0}^{n-1} z^k$ by $z - 1$ gives $z^n - 1 = 0$.

Problem 13 (2009 AIME II Problem 8 ©MAA)**.**
Dave rolls a fair six-sided die until he gets a six for the first time. Independently, Linda rolls a fair six-sided die until she gets a six for the first time. Find the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die.

Solution: Let D be the number of rolls for Dave and L be the number of rolls for Linda. By the law of total probability,

$$
\begin{aligned}
\mathbb{P}(|D-L| \leq 1) \;&=\; \mathbb{P}(1 \leq D \leq 2)\mathbb{P}(L=1) + \sum_{\ell=2}^{\infty} \mathbb{P}(\ell - 1 \leq D \leq \ell + 1)\mathbb{P}(L=\ell) \\
&= \left(\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6}\right)\left(\frac{1}{6}\right) + \sum_{\ell=2}^{\infty}\left[\left(\frac{5}{6}\right)^{\ell-2}\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^{\ell-1}\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^{\ell}\left(\frac{1}{6}\right)\right]\left(\frac{5}{6}\right)^{\ell-1}\left(\frac{1}{6}\right) \\
&= \frac{1}{36}\left[\frac{11}{6} + \sum_{\ell=2}^{\infty}\left(\frac{5}{6}\right)^{2\ell-3} + \left(\frac{5}{6}\right)^{2\ell-2} + \left(\frac{5}{6}\right)^{2\ell-1}\right] \\
&= \frac{1}{36}\left[\frac{11}{6} + \left(\frac{5}{6}\right)^{-3}\sum_{\ell=2}^{\infty}\left(\frac{25}{36}\right)^{\ell} + \left(\frac{5}{6}\right)^{-2}\sum_{\ell=2}^{\infty}\left(\frac{25}{36}\right)^{\ell} + \left(\frac{5}{6}\right)^{-1}\sum_{\ell=2}^{\infty}\left(\frac{25}{36}\right)^{\ell}\right] \\
&= \frac{1}{36}\left[\frac{11}{6} + \frac{6^3}{5^3}\cdot\frac{25^2}{36}\cdot\frac{1}{11} + \frac{6^2}{5^2}\cdot\frac{25^2}{36}\cdot\frac{1}{11} + \frac{6}{5}\cdot\frac{25^2}{36}\cdot\frac{1}{11}\right] \\
&= \frac{1}{36}\left[\frac{11}{6} + \frac{30}{11} + \frac{25}{11} + \frac{125}{66}\right] \;=\; \frac{8}{33}.
\end{aligned}
$$