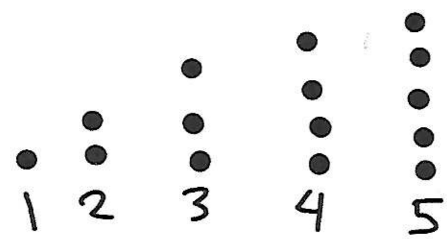
SUCCESSIVE DIFFERENCES

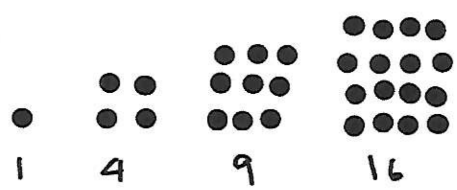
BEGINNER CIRCLE 2/10/2013

1. Shape Numbers

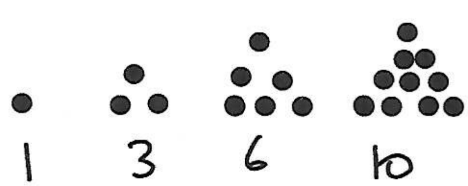
We all know about the numbers. But what about the numbers that arise from looking at different shapes. For instance we have the line numbers:



which are easy to compute– the nth line number is just n. We also have the square numbers



which are also fairly easy to compute: the nth square number is just n^2. But what about the triangular numbers



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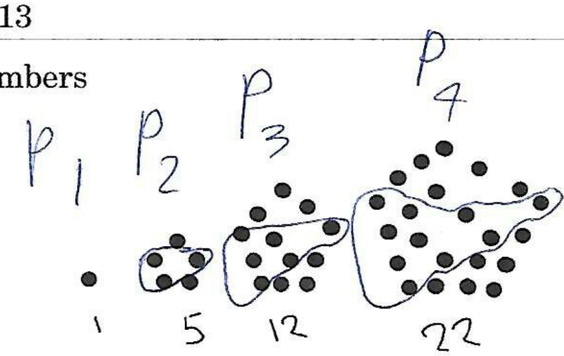
or the pentagonal numbers



and so on? Can you find closed formulas for these?

Problem 1. Using the formula for the area of a triangle, find an approximation for the triangular numbers. Can you make this approximation better? Explain your method in full sentences.

Area of a triangle is base times height, divided by 2 so we find n dots at the bottom and n layers so we guess about $\frac{n^2}{2}$. This is slightly below, so try adding 1 to the height and see what happens.

Problem 2. What is the difference between the 4th triangular number and the 5th triangular number. What about the 5th and 6th? What about the nth and $n+1$th? Why?

$T_5 - T_4 = 5 \quad T_6 - T_5 = 6 \quad T_{n+1} - T_n = n+1$

This is because the n^{th} triangulan number adds n dots to the bottom of the triangle.

Problem 3. Using the previous problem as inspiration, write each triangular number as the sum of smaller line numbers. Explain why your formulation works in full sentences.

$$T_n = L_n + L_{n-1} + \ldots + L_2 + L_1 = \sum_{i=1}^{n} L_i$$

This works because the triangle is just stacked line numbers from 1 to n.

Problem 4. Take each pentagon in the pentagonal numbers and break it down into several smaller triangles. Can you represent the pentagonal numbers as a sum of triangular numbers?

If we look at the middle of the P_n Pentagon, We can outline a triangular number T_n, then the outsides are two T_{n-1} numbers.

Finally, we fing $\boxed{P_n = T_n + 2T_{n-1}}$

Problem 5. What is the difference between the 4th and 5th pentagonal numbers? What about the 5th and 6th? Can you find a pattern? Explain in full sentences

$P_5 - P_4 = T_5 + 2T_4 - (T_4 + 2T_3) = \underset{15}{T_5} + \underset{10}{T_4} - 2\underset{6}{T_3} = \boxed{13}$

$P_6 - P_5 = \underset{21}{T_6} + \underset{15}{T_5} - 2\underset{10}{T_4} = 36 - 20 = \boxed{16}$

$P_4 - P_3 = 10$

$P_3 - P_2 = 7 \quad P_2 - P_1 = 4 \quad P_1 - 0 = 1$

Each difference seems to increase by 3. So $\boxed{P_{n+1} - P_n = 3n+1}$

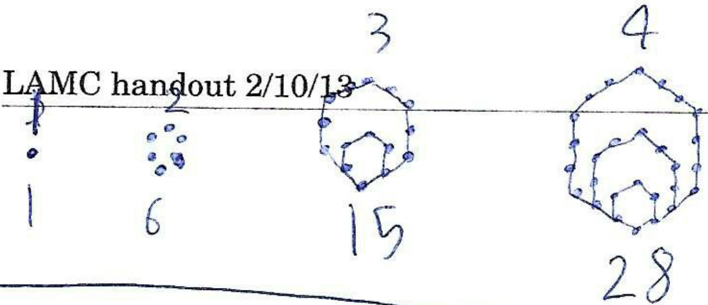
Problem 6. Using your pattern in the problem above, find all of the pentagonal numbers up to the 8th pentagonal number.

$P_4 = 22 \quad P_5 - P_4 = 3 \cdot 4 + 1 \Rightarrow \boxed{P_5 = 35}$

$\boxed{P_6 = P_5 + 3 \cdot 5 + 1 = 51}$ $\boxed{P_7 = P_6 + 3 \cdot 6 + 1 = 70}$

$\boxed{P_8 = P_7 + 3 \cdot 7 + 1 = 92}$

Problem 7. Draw a few pictures of what the hexagonal numbers should look like. Use the same methods as in the previous 3 problems to find all of the hexagonal numbers up to the 6th hexagonal number.



$H_4 - H_3 = 13$

$H_3 - H_2 = 9$

$H_2 - H_1 = 5$

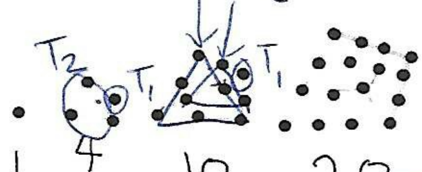
$H_1 - 0 = 1$

$H_{n+1} - H_n = 4 \cdot n + 1$

$\boxed{H_5 = H_4 + 4 \cdot 4 + 1 = 45}$

$\boxed{H_6 = H_5 + 4 \cdot 5 + 1 = 66}$

Problem 8. The tetrahedral numbers are given by little pyramids with triangular bases. Can you write each tetragonal number as a sum of triangular numbers? Write an explanation in full sentences.



$$Tet_n = T_1 + T_2 + \ldots + T_{n-1} + T_n = \boxed{\sum_{i=1}^{n} T_i}$$

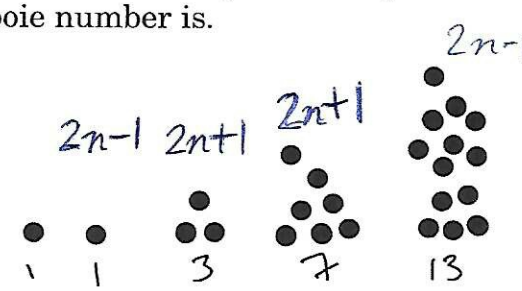
Problem 9. Using the problem above as motivation, find all of the tetrahedral numbers up to the 7th one.

$\boxed{Tet_5 = Tet_4 + T_5 = 35}$

$\boxed{Tet_6 = Tet_5 + T_6 = 56}$

$\boxed{Tet_7 = Tet_6 + T_7 = 56 + 28 = 84}$

Problem 10 (Bonus)**.** The Gooey Kablooie numbers are given by the pictures below. Use the techniques that you have developed in the previous 5 problems to figure out what the 8th Gooey Kablooie number is.



$\cancel{G_1 - 0 = 1}$ $G_2 - G_1 = 0$ $G_3 - G_2 = 2$ $G_4 - G_3 = 4$

$G_5 - G_4 = 6$ $G_{n+1} - G_n = 2\cdot(n-1)$

So $\cancel{G_6}$ $\boxed{G_6 = 8 + G_5 = 21}$

$\boxed{G_7 = G_6 + 10 = 31}$

$\boxed{G_8 = G_7 + 12 = 43}$

2. Successive Differences

In the warm up , we looked at sequences of numbers and tried to figure out the pattern to the sequence. Sometimes, it is really easy for us to figure out the pattern: for example, we all know how to figure out every number in the pattern

$$2, 4, 6, 8, \ldots$$

In fact, we can do even better than just figure out every number in the pattern, we can provide a formula that computes the nth number

$$2n$$

One way to identify a pattern is to look at the successive differences of the numbers in the pattern. Let's give some language to describe these patterns.

Definition 1. A **sequence** is a bunch of numbers that come in a particular order. When we talk about the whole sequence, we will use a capital letter. For example the sequence of even numbers might be written as

$$E = 2, 4, 6, 8$$

When we want to refer to a specific part of this sequence, we will write the part of the sequence that we want to talk about using lowercase letters. For example,

$$e_1 = 2$$
$$e_5 = 10$$
$$e_n = 2n$$

Definition 2. If we are given a sequence

$$A = a_1, a_2, a_3, \ldots$$

we define the **difference sequence of** A (which we will write dA) to be the sequences of all the differences of A. For example, if my sequence A is

$$A = 2, 3, 5, 8, 9, 9, 9, 9, \ldots$$

then

$$dA = 1, 2, 3, 1, 0, 0, 0, \ldots$$

Formally, the sequence dA is given by entries

$$b_i = a_{i+1} - a_i$$

Problem 11. Let N be the sequence of numbers,

$$N = 1, 2, 3, 4, \ldots$$

. What is dN?

$dN = 1, 1, 1, 1, 1, 1 \ldots$

Problem 12. Let S be the sequence of square numbers,

$$S = 1, 4, 9, 16, \ldots$$

What is dS?

$dS = 3, 5, 7, 9, \ldots$

Problem 13. Find all the sequences A have the property that

$$dA = 0, 0, 0, 0, \ldots$$

Any $A = c, c, c, c, c, \ldots$ where c is a "fixed" element of the real numbers.

Notation: $c \in \mathbb{R}$

Problem 14. Frequently, we will want to take the difference of a sequence several times. Let T be the sequence of triangular numbers (from the warm-up.) What is dT? What about ddT?

$dT = 2, 3, 4, 5, \ldots$

$ddT = 1, 1, 1, 1, \ldots$

Problem 15. Let A be a sequence. Suppose that I know that $a_1 = 0$, and I know that $dA = T$, where T are the triangular numbers. Can you find the a_5, the fifth entry of the sequence A?

~~$a_i = T_i + a_{i}$~~ $a_i - a_{i-1} = T_{i-1}$ $dA = 1, 3, 6, 10, 15, \ldots$

$a_2 = T_1 + a_1 = 1$ $a_3 = T_2 + a_2 = 4$ $a_4 = T_3 + a_3 = 10$

$\boxed{a_5 = T_4 + a_4 = 20}$

Problem 16. Let S be the sequence of square numbers. Prove that $dS = O$, where O is the sequence of odd numbers.

$S = \underset{a_1}{1}, \underset{a_2}{4}, \underset{a_3}{9}, \underset{a_4}{16}, \underset{a_5}{25}, \ldots$ $dS = \underset{b_1}{3}, \underset{b_2}{5}, \underset{b_3}{7}, \underset{b_4}{9}, \ldots$

Let's take $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$

$2n+1$ is the $(n+1)^{th}$ odd number, so we get

$dS = O$ excluding 1, or $\boxed{dS = O \setminus \{1\}}$

$\boxed{b_i = 2i + 1}$

Problem 17. We've noticed that with the triangular numbers T, that $dddT = 0$, and with the square numbers, $dddS = 0$ (Where 0 means the sequence of all zeroes.) Show that the sequence of cubic numbers C have the property that

$$d^4 = ddddC = 0$$

(Hint: There is a faster way that computing $ddddC$)

$(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1$

$dC_i = 3i^2 + 3i + 1$ Now take ddC as

$\begin{pmatrix} 3(n+1)^2 + 3(n+1) + 1 \\ - (3n^2 + 3n + 1) \end{pmatrix} = \begin{pmatrix} 3n^2 + 6n + 3 + 3n + 3 + 1 \\ - (3n^2 + 3n + 1) \end{pmatrix} = 6n + 6$

$ddC_i = 6i + 6$ d^3C is $6(n+1) + 6 - (6n+6) = 6 = d^3C_i$

$\boxed{d^4C = 0, 0, 0, \ldots}$

Problem 18. Can you find a (non-zero) sequence such that $dF = F$?

$a_i = 2^{i-1}$ has $b_i = a_{i+1} - a_i = 2^i - 2^{i-1} = 2^{i-1}(2-1)$

$= 2^{i-1}(1) = 2^{i-1} = a_i$

$\boxed{F = 1, 2, 4, 8, \ldots}$

Problem 19. Can you find a (non-zero) sequence where $ddF = F$?

$\boxed{F = 1, 2, 4, 8, \ldots}$

Problem 20. Find a (non-zero) sequence that has the property that

$dF = F$with all the numbers shifted to the right by one place

$\boxed{F = 1, 1, 2, 3, 5, \ldots}$ $\boxed{\text{Fibonacci Sequence}}$

$\boxed{dF = 0, 1, 1, 2, 3, \ldots}$ We allow 0 to take place of empty spot

3. Polynomials

First off, what is a polynomial?

Definition 3. A **polynomial** $f(n)$ is an equation made multiplying and adding together numbers, and powers some variable n. For instance, the following are all polynomials:

$$n^2 + 2; \qquad 22n + 34n^3 5 \qquad \frac{347928}{2837} \qquad (n^2+1)(n+5)$$

Definition 4. The **degree** of a polynomial $f(n)$ is the highest exponent that appears in the polynomial when the polynomial is multiplied out. For instance

$$n^3 + 2n + 6$$

has degree 3, while the polynomial

$$(n^4 + 2n)(n^2 + 1)$$

has degree 6

One convenient way to make sequences is with polynomials. For example we have already seen the sequence of squares, given by

$$s_n = n^2$$

and the sequence of triangular numbers, which is given by

$$t_n = \frac{1}{2}n^2 + \frac{1}{2}n$$

Our question is, which sequences are given by a polynomial?

Problem 21. Find 3 polynomials that give the sequences of Odd (0), Even (E), and Threeven (S) numbers respectively:

$$O = 1, 3, 5, 7, 9, \ldots$$
$$E = 2, 4, 6, 8, 10, \ldots$$
$$P = 3, 6, 9, 12, \ldots$$

$\boxed{O_n = 2n - 1}$ $\boxed{E_n = 2n}$ $\boxed{P_n = 3n}$

All of first degree power since differences are constants

Problem 22. Can you find a polynomial that fits the sequence

$$A = 1, 0, 1, 4, \ldots$$

NO

Problem 23. Can you find a polynomial that fits the sequence

$$A = 1, -1, 1, 15, \ldots$$

NO

As you can see, it is quite hard to find the polynomial that fits a certain number of points! However, it is quite easy to find out sequences by looking at differences.

Problem 24. Let A and B be two different sequences. Define the **sum sequence** to be the sequence where each of their entries is summed together. For example of the sequences

$$A = 1, 3, 2, 5, 4, \ldots$$
$$B = 1, 1, 2, 2, 3, \ldots$$
$$A + B = 2, 4, 4, 7, 7, \ldots$$

Can you show that

$$d(A + B) = dA + dB$$

$dA = 2, -1, 3, -1, \ldots$ $dB = 0, 1, 0, 1, \ldots$

$d(A+B) = 2, 0, 3, 0$

$dA + dB = 2+0, -1+1, 3+0, -1+1 = 2, 0, 3, 0$

Problem 25. Let P be a sequence given by a polynomial of degree 2 (that is,

$$p_n = f(n) = an^2 + bn + c$$

for some quadratic polynomial n). Show that dP is given by a polynomial of degree 1.

$$dP_n = a(n+1)^2 + b(n+1) + c - (an^2 + bn + c)$$
$$= \boxed{2an + a + b} \leftarrow \text{a first deghee polynomial.}$$

Problem 26. Let P be a sequence given by a polynomial of degree k. Explain in a few sentences why dP is given by a polynomial of degree $k-1$.

Because the highest degree term always has a coefficient a 1 when it is summed with one and taken to a powere

Problem 27. Let P be a sequence given by a polynomial of degree n. Show that

$$d^{n+1}P = ddd\dots dddP = 0,$$

where 0 is the sequence of all 0.

Induction: Base Case: Degree 1 $an+b = P$

$dP = b, b, \dots$ $d^2P = 0$. So we find

$d^{n+1}P(\deg n) = 0$, look at $d^{n+2}P(\deg n+1) = d\,P(\deg 0)$

$= d\left[d^{n+1}P(\deg n+1)\right] = 0.$

Problem 28. Suppose I know that the sequence P starts as

$$2, 5, 8$$

and I know that

$$ddP = 0$$

Can you find the rest of the entries of P

$P = 2, 5, 8, 11, 14, 17, \dots$

$P_n = 3n - 1$