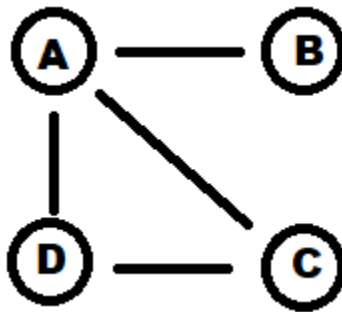


GRAPH THEORY: AN INTRODUCTION

BEGINNERS 3/4/2018

1. GRAPHS AND THEIR PROPERTIES

A **graph** G consists of two sets: a set of vertices V , and a set of edges E . A vertex is simply a labeled point. An edge is a connection between two vertices. For example, suppose we have vertex set $V = \{a, b, c, d\}$ and edges $E = \{(a, b), (a, c), (a, d), (c, d)\}$. We can see a visual representation of this graph $G = (V, E)$ with 4 vertices and 4 edges below:



Our first graph: $G = (V, E)$

Graphs can represent all sorts of different real life situations like internet router connections, social networks, even links between websites (this is how Google calculates their webpage rankings!). We will think of the nodes as representing friends at a dinner party, and edges representing a handshake occurring between those two friends.

As with any mathematical object, we would like to define some *properties* of a graph so that we can study particular graphs more deeply. Since graphs can be arbitrarily large, it is easier to begin our focus on so-called *local quantities* of the graph. That is, we will look at each small part of the graph and use these observations to conclude something about the graph as a whole.

Recall, each edge is associated to two vertices. For edge $e = (a, c)$, we say e is **incident to** vertices a and c . We also say that vertices a and c are **adjacent**, as they are incident to a common edge. For a vertex v in the vertex set V , we define the **degree** of v , written $\deg(v)$, to be **the total number of edges incident to vertex v** . In our example, the degree of a vertex corresponds to the number of handshakes that person has participated in.

(1) Calculate the degree of each vertex in the graph G .

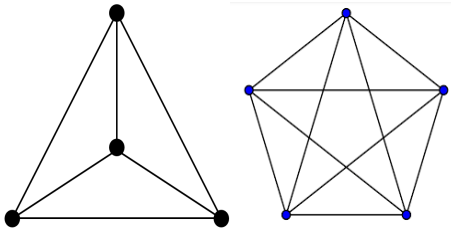
(a) $\deg(a) = \underline{\hspace{2cm}}$

(b) $\deg(b) = \underline{\hspace{2cm}}$

(c) $\deg(c) = \underline{\hspace{2cm}}$

(d) $\deg(d) = \underline{\hspace{2cm}}$

(2) We say a graph is **complete** if there exists an edge between every pair of vertices. Shown below are the graphs K_4 (the complete graph on four vertices) and K_5 (the complete graph on 5 vertices).



(a) What is the degree of each vertex in K_4 ?

(b) What is the degree of each vertex in K_5 ?

(c) What will be the degree of each vertex in K_n ? Justify your answer.

(3) We say a vertex v is **even** if $\deg(v)$ is even. Similarly, v is **odd** if $\deg(v)$ is odd.

(a) How many even vertices are there in the first graph, G ?

(b) How many odd vertices are in the graph G ?

(c) For what values of n will each vertex of K_n have even degree? Justify your answer.

(d) For what values of n will each vertex of K_n have odd degree? Justify your answer.

(4) Draw your own graph $G_2 = (V_2, E_2)$ below. Next to your graph, define the vertex set V_2 and the edge set E_2 . You must draw at least 4 vertices and at least 4 edges. Keep track of the degree of each vertex in order to answer the questions on the following page.

- (a) How many even vertices are in G_2 ?
- (b) How many odd vertices are in G_2 ?
- (c) Examine your neighbor's graphs and their answers for part (b). Make a hypothesis about the number of odd vertices in an arbitrary graph.
- (d) Draw more graphs below to either affirm or contradict your hypothesis. Continue until you are confident in your answer.

(5) After playing with a few more graphs, you should notice that the number of odd vertices seems to always be even. This turns out to be true and is famous enough to have a name: the **Handshaking lemma**. In terms of our example, the lemma says in a party of people, some of whom shake hands, an *even* number of people must shake an odd number of hands. This fact was first proven by Euler many centuries ago, let's see how! Suppose we have a graph $G = (V, E)$ consisting of n vertices and m edges.

(a) Let e be an edge in E . How many vertices are incident to e ? Note: each point where an edge meets a vertex contributes 1 to the degree of that vertex.

(b) Using your answer to (a), argue that if we add together the degree of every vertex, the result will be even. (Bonus: what will the sum be exactly?)

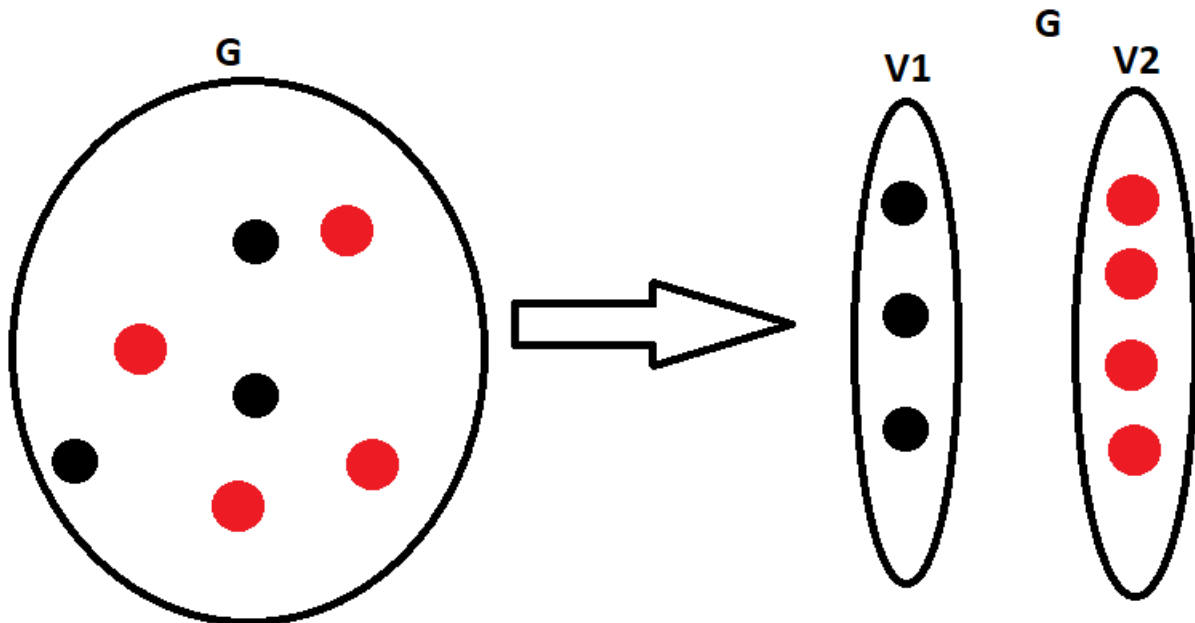
(c) If we add together the degree of every even vertex, will the result be even or odd?

(d) Use (b) and (c) to argue that if we add together the degree of every odd vertex, the result must be even.

(e) Use (d) to argue that the total number of odd vertices is even.

2. BIPARTITE GRAPHS

Bipartite Graphs are special kinds of graphs that follow a few rules. Let us take an edgeless graph G such as shown below with vertices in the set V . Now, split the vertices into two different sets V_1 and V_2 such that any vertex v is either in V_1 or V_2 , **not both** of them. Now, draw an edge connecting any black vertex in V_1 **only** to any red vertex in V_2 . Repeat this process as many times as you like. You have just constructed a bipartite graph.



The restrictions on a bipartite graph follow from the process we just went through. Basically, a bipartite graph can **always** have its vertices separated into two **disjoint** sets such that every vertex is only adjacent to vertices in the other set. Two disjoint sets have **no** elements in common. Below, the restrictions are represented in their symbolic and more precise mathematical form, which you should try to understand as it is the formal language of mathematics. The symbol $A \cup B$ represents the union of sets A and B (that is, the set you get by combining all the elements in A and B into a single set). The symbol $A \cap B$ represents the intersection (that is, the set of all elements that are both in A and B).

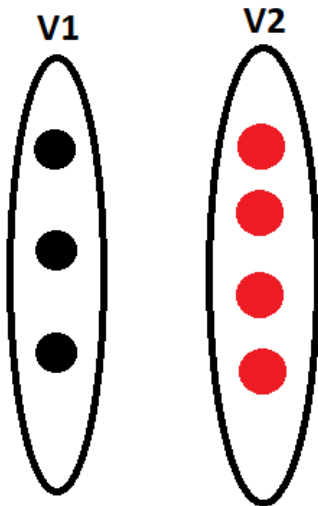
Given a graph B with vertices in V , B is bipartite if there exist sets of its vertices V_1 and V_2 such that:

- (1) If vertex $v \in V_1$, then it may only be adjacent to vertices in V_2 .
- (2) If vertex $v \in V_2$, then it may only be adjacent to vertices in V_1 .
- (3) $V_1 \cap V_2 = \emptyset$
- (4) $V_1 \cup V_2 = V$

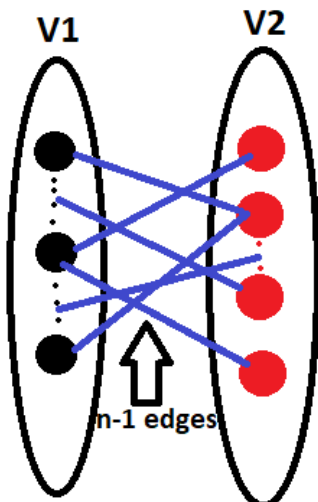
If V_1 and V_2 are the disjoint sets that satisfy the bipartiteness requirements for a graph B , then V_1 and V_2 are called the **bipartition** of B . Now, we are going to prove a few lemmas and theorems about bipartite graphs, ultimately culminating in the Bipartite Coloring Theorem.

(1) **Bipartition Equal Degree Theorem:** Given a bipartite graph B and bipartition V_1 and V_2 , the sum of the degrees of all the vertices in V_1 is equal to the sum of the degrees of all the vertices in V_2 .

(a) Let us take the edgeless graph we used at the beginning of this section. Draw a single edge so that the graph remains bipartite. Show that the Bipartition Equal Degree Theorem holds in this case.



(b) Now, assume that the Bipartition Equal Degree Theorem holds for the $n - 1$ edge graph below. Draw a single edge that maintains its bipartiteness, and show that the Bipartition Equal Degree Theorem still holds.

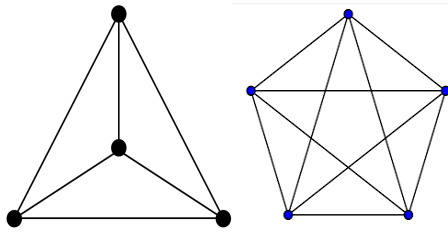


(c) Now use (a) and (b) as the base case and inductive step to form a proper proof by induction.

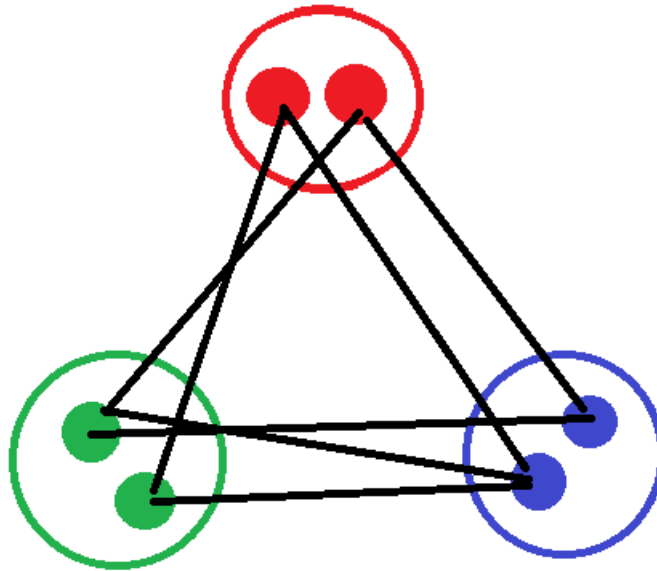
(2) **Regular Bipartite Theorem:** Similar to the K_n graphs, a k -regular graph G is one where every vertex $v \in V(G)$ has $\deg(v) = k$. Now, using problem 1, show that every k -regular, bipartite graph B has the same number of vertices in either set of its V_1 and V_2 bipartition.

(3) **Graph Coloring:** We say a graph G can be colored by splitting its vertices into colored sets such that no two vertices of the same color are adjacent to each other (so if two vertices are in the green set, they can only be adjacent to vertices in other colors). If n is the smallest number of sets that the vertices of graph G can be split into so that G is properly colored, then G is an n -colorable graph. Solve the coloring problems below.

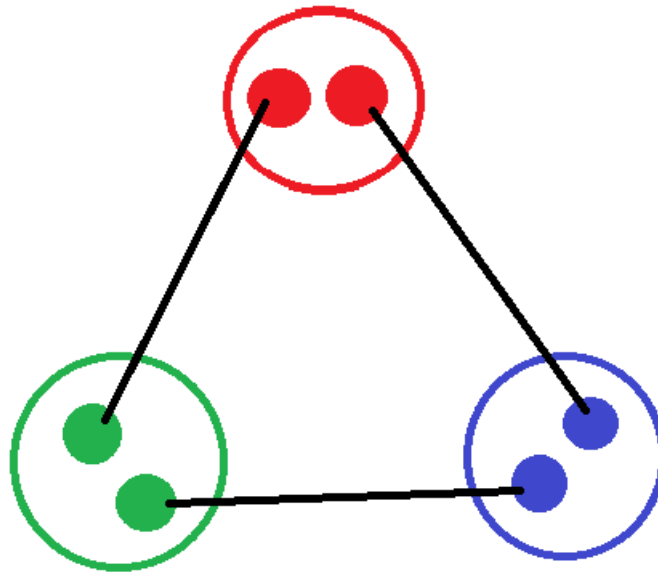
(a) What is the coloring of K_4 and K_5 ? As you can see, all graphs with n vertices are trivially colorable by n different colors. That's why graph theorists try to find the minimum number of colors k needed to color a graph and then label that graph k -colorable.



(b) What is the coloring of the below graph?



(c) Is the below graph 3-colorable?



(d) What is the coloring of a graph that is composed of the vertices and edges of an $2n$ -sided polygon?

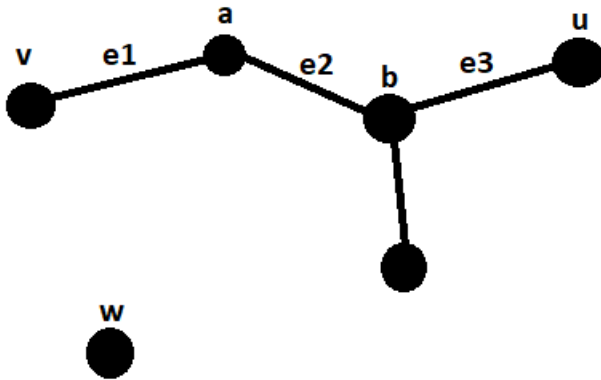
(4) **Bipartiteness Coloring Theorem:** As long as a bipartite graph B is not edgeless, then it is 2-colorable.

(a) Draw a non-edgeless bipartite graph.

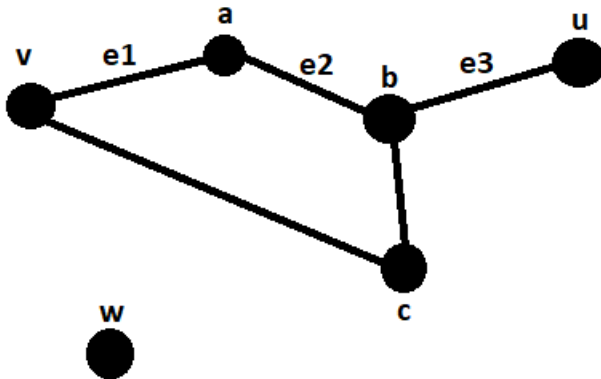
-
- (b) Choose two sets of vertices so that each set satisfies the color restriction.
- (c) How many vertices of your graph must be in a single color set for it to be considered 1-colorable?
- (d) Why does a single color set violate the coloring restriction for your graph?
- (e) What type of graph must G be in order for it to be 1-colorable

3. PATHS AND EULERIAN CYCLES

A path along a graph G is a sequence of alternating edges and vertices such that each edge is incident to the vertices it is next to in the sequence and each edge and vertex in the sequence is distinct. For the graph below, a path between vertex v and u can be written as $v, e_1, a, e_2, b, e_3, u$. However, since no edges are incident to w , no paths can be written from w to v or any other vertex in the graph.



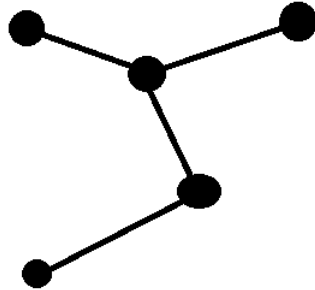
A cycle along a graph G is a path that ends at the same vertex that it started at. The length of a path is the number of edges in the path's sequence. Modifying the graph above gives a cycle of length 4 at vertices v , a , b , and c .



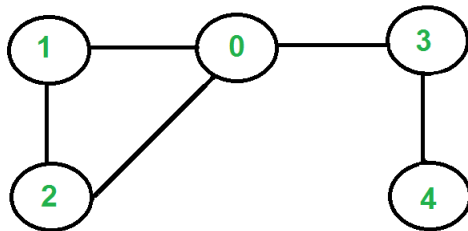
-
- (1) **Bipartite Cycles:** Show that there are no odd cycles in a bipartite graph B with bipartition X and Y .
- (a) Pick an arbitrary vertex in either X or Y and assume it is part of an odd cycle. Write down the general sequence that would represent this path, numbering each edge in order (that is, the first edge in the sequence is e_1 , the second is e_2 , etc.)
- (b) What is the parity of the last edge in the sequence?
- (c) Do vertices in X precede every odd edge in the cycle? Which vertices do an odd edge lead to?
- (d) Use (b) and (c) to show that an odd cycle in a bipartite graph provides a contradiction, and thus is impossible.

- (2) **Eulerian Paths:** An Eulerian path is a path that visits every edge in a graph exactly once. An interesting problem for graph theorists is whether certain graphs have Eulerian paths since not all graphs do. Find the Eulerian paths of the following graphs. If you cannot, explain why.

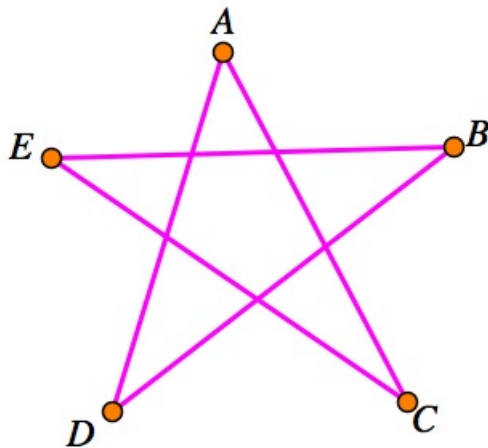
(a) .



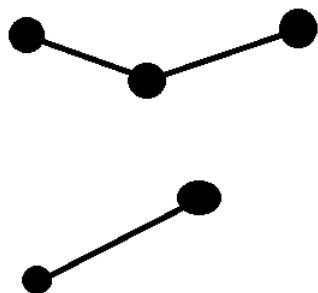
(b) .



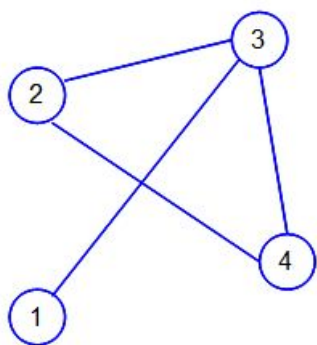
(c) .



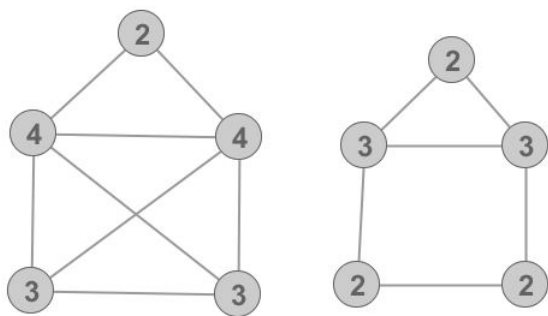
(d) .



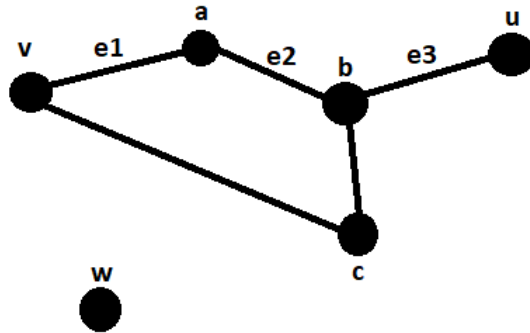
(e) .



(f) .

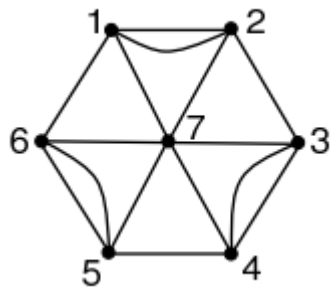


(g) .

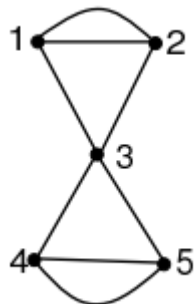


(3) **Eulerian Cycles:** An Eulerian cycle is a cycle that visits every edge in a graph exactly once. Find the Eulerian cycles of the following graphs. If you cannot, explain why.

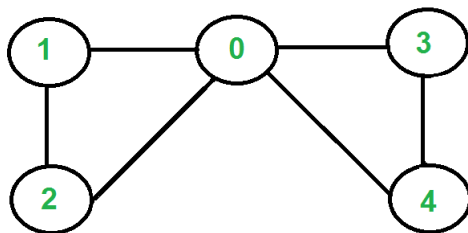
(a) .



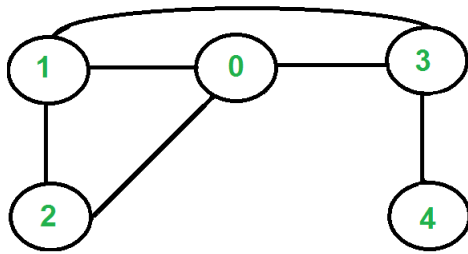
(b) .



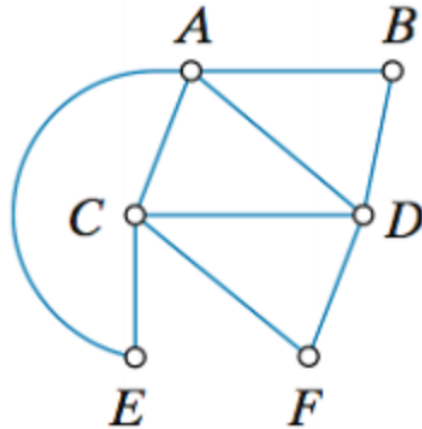
(c) .



(d) .



(e) .



(4) **Eulerian Paths and Odd Degree Vertices:** Knowing the number of vertices with odd degrees can actually determine whether an Eulerian path exists before you even attempt to draw it out!

(a) Other than the starting and ending vertices in an Eulerian path, each **intermediate** vertex in the path must be traveled to by one edge and away from another. So what is the parity of the number of edges that are incident upon each intermediate vertex?

(b) What is the parity of the degree of each intermediate vertex in the Eulerian path?

-
- (c) How many odd degree vertices can be in a graph with an Eulerian path?
- (d) If a graph with an Eulerian path contains an odd vertex, can this path ever end on the starting vertex in an Eulerian cycle? (**Hint:** What do you know about the starting vertex if it is visited as an intermediate vertex? How many edges are required to leave a vertex and return to it in an Eulerian path?)