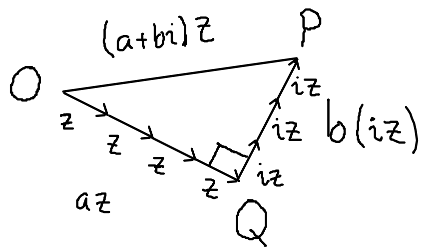
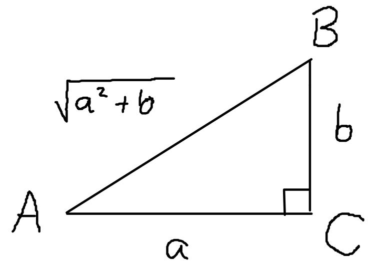
Complex Numbers and Geometry

Review of the Complex Plane

Recall that we think of the complex numbers as a plane, identifying the complex number $a+bi$ with the point written in Cartesian coordinates as (a, b). We can also identify complex numbers by the distance r from 0 in the complex plane, and the angle θ made with the horizontal axis. The picture below illustrates multiplying a complex number z by the complex number $a+bi$.





$$(a+bi)z = az + i(bz)$$

In the picture above, triangle OPQ is similar to triangle ABC, so $\angle O = \angle A$ and we have

$$\frac{OP}{OQ} = \frac{AB}{AC}.$$

Therefore,

$$OP = \frac{AB}{AC} OQ = \frac{\sqrt{a^2+b^2}}{a} a|z| = \sqrt{a^2+b^2}|z| = r|z|.$$

Therefore $(a+bi)z$ makes an angle of θ with z, and is r times longer than z. We say that the effect of multipyling by $a+bi$ is to stretch by a factor of r, and rotate by an angle of θ (an "amplitwist").

Complex Numbers and Transformations

Try to write each of the following transformations as a complex mapping.

Words to Formulas

1. Rotation by 90 degrees counterclockwise clockwise about the origin.

   Solution: $z \mapsto iz$.

2. Rotation by 90 degrees clockwise about the origin.

3. Relfection in the horizontal axis.

4. Reflection in the vertical axis. (Hint: Try to represent the reflection as a composition of the operations in the first three problems)

5. Reflection over a line through the origin at a 45 degree angle with the horizontal axis.

6. Rotation by 90 degrees about the point $(2, 3)$.

Formulas to Pictures

Draw a picture of each of the mappings given by each of the following formulas:

1. $z \mapsto iz$

2. $z \mapsto -iz$

3. $z \mapsto i(z+1) - 1$

4. $z \mapsto \frac{1}{i}(iz+1)$

Pictures to Formulas

For each of the following, try to represent the transformation as a formula for a complex mapping:

1.

2.

3.

4.

Formulas to Words

For each of the following, give a verbal description of the transformation described.

1. $z \mapsto -z$

2. $z \mapsto -iz$

3. $z \mapsto \frac{1}{1+i}(z - i)$

4. $z \mapsto 2z + 1$

5. (*) $z \mapsto \frac{1}{z}$.